6th May 20

Super Manifolds : Motivation and Introduction

Motivation

Free bosons obey a canonical commutation relations

$[a, b] := ab - ba = i\hbar$

In the classical limit $\hbar \to 0$

a, b can be represented by real or complex numbers

Free fermions obey canonical anticommutation relations

$\{\psi, \chi\} := \psi\chi + \chi\psi = i\hbar$

In the classical limit $\hbar \to 0$, ψ, χ must be zero.

We take quasi-classical limit ψ, χ as Grassman algebra valued.

"The quasi-classical limit of a QFT requires Grassman algebra valued fields"

* Also if we have ghosts, antifields etc. (BV-BRST) require Grassman algebra valued field.

Remark The name "Super" ($\mathbb{Z}_2$-graded) comes supersymmetry

Bosons $\rightleftarrows$ Fermions

and supermanifolds are very important in supersymmetry

however, supermanifold are well-motivated without supersymmetry

Many signs in Maths are "super" in nature:

Cohomology, homological algebra, differential forms etc.

Grassmann algebras (1844)

is an $\mathbb{R}$ - algebra generated by $\{\theta^1, \theta^2, \cdots \theta^m\}$

Subject to $\theta^i \theta^j = -\theta^j \theta^i$, $(\theta^i)^2 = 0$

we have a polynomial algebra

"supermaths before supermaths"

$$\mathbb{R}[\theta^1, \theta^2] \ni f(\theta^1, \theta^2) = f_0 + \theta^1 f_1 + \theta^2 f_2 + \theta^1 \theta^2 f_{21}$$

with $f_0, f_1, f_2, f_{21} \in \mathbb{R}$

$\mathbb{Z}_2$ - graded by assign degree 1 to θ's

$f_0 + \theta^1 \theta^2 f_{21}$ is degree 0 or even

$\theta^1 f_2 + \theta^2 f_2$ is degree 1 or odd

with the obvious generalisation to $\mathbb{R}[\theta] := \mathbb{R}[\theta^1, \dots, \theta^m]$

Notation Denote the degree or parity of f

$\tilde{f} \in \mathbb{Z}_2$

$$fg = (-1)^{\tilde{f}\tilde{g}} gf \quad , \text{ for homogeneous } f, g.$$

II) Supermanifolds

The formal definition is in terms of locally ringed spaces, i.e. using algebraic geometry. (Berezin + Leites 1975)

Can we can skip details using local coordinates

"Supermanifolds = Manifold with commuting and anticommuting coordinates"

A superdomain $\mathcal{U}^{p|q}$ has the underlying topological space

$\mathcal{U}^p \subseteq \mathbb{R}^p$ (open) and the algebra is

is $C^\infty(\mathcal{U}^p) \otimes \mathbb{R}[\theta^1, \dots, \theta^q]$

"deform" C^∞ with a Grassman algebra.

~~the the patch togethe~~

We then use coordinates $x^A = (x^a, \theta^\alpha)$

form coordinates on U^p

Commutation relations

$$x^a x^b = x^b x^a$$
$$x^a \theta^\alpha = \theta^\alpha x^a$$
$$\theta^\alpha \theta^\beta = -\theta^\beta \theta^\alpha$$

$$\Rightarrow \quad \theta^2 = 0$$

$\boxed{\text{Supermanifolds are built from gluing superdomains.}}$

The gluing is done using coordinates

$U^{p|q}$ $\quad x^A = (x^a, \theta^\alpha)$

$V^{p|q}$ $\quad y^B = (y^b, \eta^\beta)$

$$\Phi : U^{p|q} \longrightarrow V^{p|q}$$

is defined by pullbacks of the coordinates

$$\Phi^* : C^\infty(V^p) \otimes \mathbb{R}[\eta] \longrightarrow C^\infty(U^p) \otimes \mathbb{R}[\theta]$$

which is grading preserving

$$\Phi^* y^b = \phi^b(x) + \sum_{\text{EVEN}} \frac{1}{m!} \underbrace{\theta^{\alpha_1} \theta^{\alpha_2} \cdots \theta^{\alpha_m}}_{\text{Even } \# \, \theta\text{'s}} \phi^b_{\alpha_m \cdots \alpha_1}(x)$$

$$\Phi^* \eta^\beta = \sum_{\text{ODD}} \frac{1}{m!} \underbrace{\theta^{\alpha_1} \theta^{\alpha_2} \cdots \theta^{\alpha_m}}_{\text{ODD } \# \, \theta\text{'s}} \phi^\beta_{\alpha_m \cdots \alpha_1}(x)$$

We write $\quad y^B = y^B(x) \quad (x, \theta)$

Denote a supermanifold as M (we have defined via the gluing of superdomains)

<u>Note</u> underlying M is a manifold "turning off θs" denote $|M|$.

$$\boxed{\begin{array}{l} C^\infty(U^p) \otimes \cancel{\mathbb{R}}[\theta] \\ C^\infty(U^p) \end{array}}$$

Morphisms of supermanifolds

$\Phi : M \longrightarrow N$

are defined as morphisms of LRS

However we can work locally with coordinates

x^A on M and y^B on N

$$\Phi^* y^B = y^B(x)$$

Vector fields on a supermanifold, locally "look like"

$$X = X^A(x) \frac{\partial}{\partial x^A} = X^a(x,\theta) \frac{\partial}{\partial x^a} + X^\alpha(x,\theta) \frac{\partial}{\partial \theta^\alpha}$$

$$\frac{\partial \theta^\beta}{\partial \theta^\alpha} = \delta_\alpha{}^\beta, \quad \frac{\partial}{\partial \theta^\alpha}(\theta^\beta \theta^\gamma) = \frac{\partial \theta^\beta}{\partial \theta^\alpha} \theta^\gamma - \theta^\beta \frac{\partial \theta^\gamma}{\partial \theta^\alpha}$$

$X \in \mathrm{Vect}(M)$, $\tilde{X} \in \mathbb{Z}_2$

Vector fields have a Lie bracket

$$(X, Y) \mapsto [X,Y] := X \circ Y - (-1)^{\tilde{X}\tilde{Y}} Y \circ X$$

1) $[X,[Y,Z]] = [[X,Y],Z] + (-1)^{\tilde{X}\tilde{Y}} [Y,[X,Z]]$

2] $[X,Y] = -(-1)^{\tilde{X}\tilde{Y}} [Y,X]$

Note if $\tilde{X} = 1$

$$[X,X] = X \circ X + X \circ X = 2X^2$$

So, $[X,X] = 0$ is non-trivial condition.

Such odd vector fields are called homological vector fields

$X^2 = 0$ $\quad (C^\infty(M), X)$ is a cochain complex.

Aside $\tilde{X} = 1$, $[X,X] = \underset{\text{Even}}{Y}$ "Supersymmetry"

$$[X,Y] = 0$$

Application Differential Forms

Let M be a smooth manifold, ΠTM is the supermanifold built in the following way.

TM $\quad (x^a, \dot{x}^b)$ and the admissible coordinate changes

$$x^{a'} = x^{a'}(x) \qquad \dot{x}^{b'} = \dot{x}^b \left(\frac{\partial x^{b'}}{\partial x^b} \right)$$

ΠTM $\quad (\underset{\text{Even}}{x^a}, \underset{\text{odd}}{dx^b})$ $\qquad dx^{b'} = dx^b \left(\frac{\partial x^{b'}}{\partial x^b} \right)$

$$dx^a \, dx^b = - dx^b \, dx^a$$

Differential forms (inhomogeneous in form degree) as functions on ΠTM

Locally,

$$\omega = \sum \frac{1}{p!} dx^{a_1} \cdots dx^{a_p} \, \omega_{a_p \cdots a_1}(x)$$

$$\omega \omega' = (-1)^{\tilde{\omega} \tilde{\omega}'} \omega' \omega$$

$$\left(\begin{array}{c} E \\ \downarrow \\ M \end{array} \longrightarrow \begin{array}{c} \Pi E \\ \downarrow \\ \end{array} \leftarrow \text{supermanifold} \right)$$

πTM comes with a homological vector field

$d = dx^a \frac{\partial}{\partial x^a}$ which is the de Rham differential

$\Rightarrow$ supermanifolds give a different way of thinking of the de Rham ~~complex~~.

Given $X \in \mathrm{Vect}(M)$

$X \longrightarrow i_X$

$X^a \frac{\partial}{\partial x^a} \longmapsto X^a \frac{\partial}{\partial dx^a}$

$\Rightarrow$ the interior derivative

Define $\quad L_X = [d, i_X] = X^a \frac{\partial}{\partial x^a} + dx^b \frac{\partial X^a}{\partial x^b} \frac{\partial}{\partial dx^a}$

The cartan calculus

$[L_X, L_Y] = L_{[X,Y]}$

$[L_X, i_Y] = i_{[X,Y]}$

$[i_X, i_Y] = 0$

$\Rightarrow$ supermanifolds offer a nice simple picture of differential forms.

Lie algebras

$(V, [-,-])$, $\quad [e_\alpha, e_\beta] = Q^\gamma_{\alpha\beta}\, e_\gamma$

Vector space

$V = V^\alpha e_\alpha$

coordinates on the manifold V

$V^{\beta'} = V^\alpha T_\alpha{}^{\beta'}$

πV supermanifold with global coordinates

$\theta^\alpha \; (= \pi V^\alpha)$

$\mathbb{R}[\Theta]$

$$Q = \frac{1}{2}\Theta^\alpha \Theta^\beta Q^\gamma_{\beta\alpha}\frac{\partial}{\partial\Theta^\gamma}$$

Jacobi Identity $\iff$ $Q^2 = 0$

$[-,-]$

Remark Put both examples "together" we get Lie algebroids.

$(E, [-,-], \rho) \iff (\Pi E, Q)$

Lie algebroid Homological Q.