THALES

Symplectic Geometry of Heat based on Souriau Lie Group Thermodynamics and Koszul Hessian Information Geometry

Frédéric BARBARESCO
French Academy of Sciences Aymé Poirson Prize 2014
SEE Ampere Medal 2007
SEE Emeritus Member

TGSI'17 at CIRM (Centre International de Recherche Mathématique)

TOPOLOGICAL AND GEOMETRICAL STRUCTURE OF INFORMATION

2017

CIRM - MARSEILLE

AUGUST 28TH - SEPTEMBER 1ST 2017

http://forum.cs-dc.org/topic/361/tgsi2017-presentation-organisation-abstract-submission
http://forum.cs-dc.org/category/94/tgsi2017
http://forum.cs-dc.org/topic/387/tgsi2017-preliminary-program

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Geometric Science of Information, 2017: GSI'17

Site internet de GSI'17: www.gsi2017.org

GSI'17, Paris, 7th-9th November

Keynote speakers:

- Jean-Michel Bismut
- Daniel Bennequin
- Alain Trouvé
- Barbara Tumpach
- Mark Girolami

This presentation is a synthesis of 4 avant-gardistes works on Geometric structures of Information (including thermodynamics)

Jean-Louis Koszul

Fundation of Hessian Structures Of Information Geometry (Koszul-Vinberg Characteristic function, Koszul 2-form)

Roger Balian

Fundation of Quantum Information Geometry Structures (Fisher-Balian Quantum Metric)

Jean-Marie Souriau

Fundation of Structures of Lie Group Thermodynamics and Statistical Geometric Mechanics (geometric temperature and entropy, Fisher-Souriau metric)

Maurice Fréchet

Fundation of Structures for probability extension in metric spaces and Fréchet Bound (Clairaut-Legendre equation of Information Geometry)

« J'étudie! Je ne suis que le sujet du verbe étudier. Penser je n'ose. Avant de penser, il faut étudier » - Gaston Bachelard, la flamme d'une chandelle

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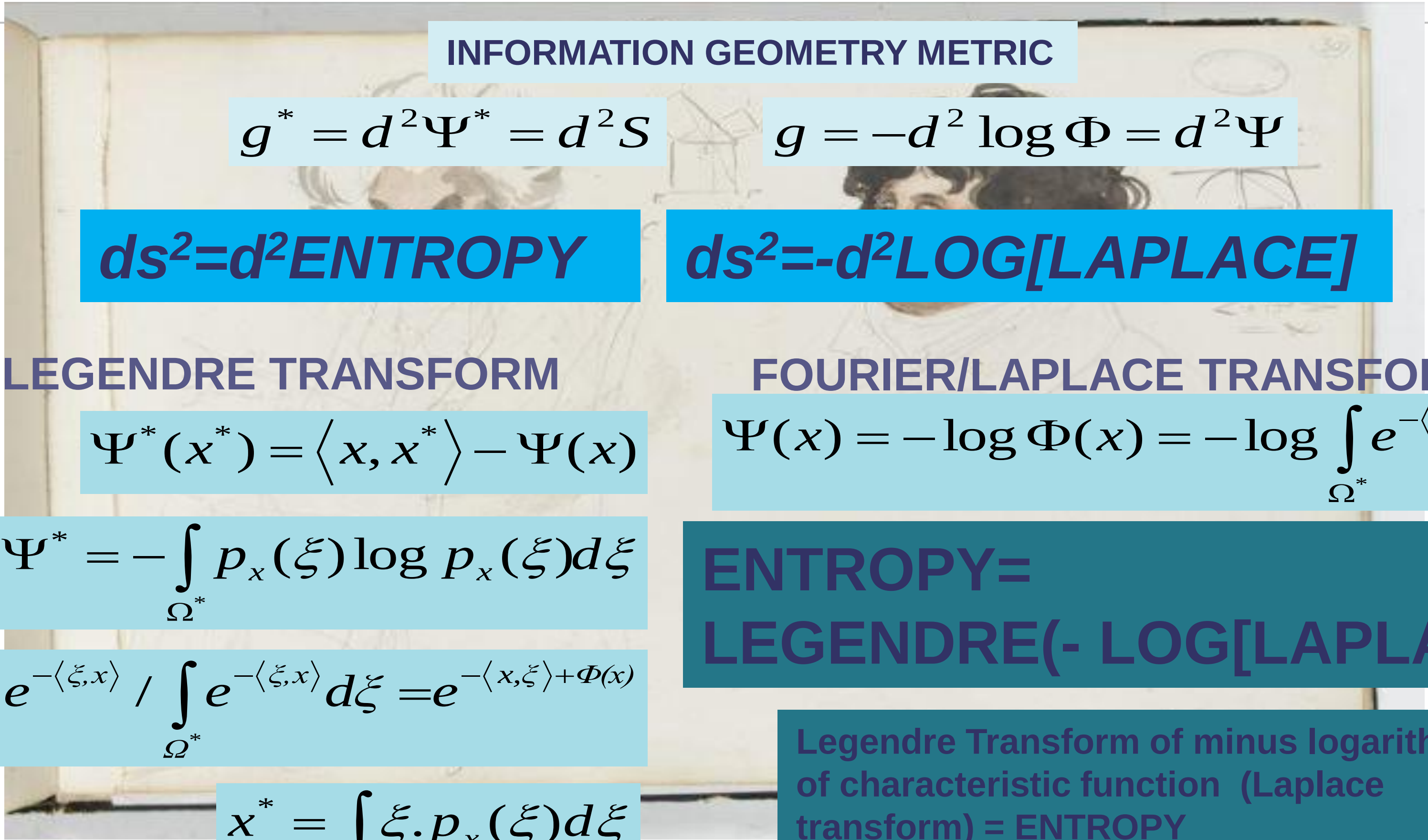
Projective Legendre Duality and Koszul Characteristic Function



INFORMATION GEOMETRY METRIC

$$g^* = d^2\Psi^* = d^2 S \qquad g = -d^2 \log \Phi = d^2 \Psi$$

ds²=d²ENTROPY

ds²=-d²LOG[LAPLACE]

LEGENDRE TRANSFORM

$$\Psi^*(x^*) = \langle x, x^* \rangle - \Psi(x)$$

$$\Psi^* = -\int_{\Omega^*} p_x(\xi) \log p_x(\xi) d\xi$$

$$p_x(\xi) = e^{-\langle \xi, x \rangle} \Big/ \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi = e^{-\langle x, \xi \rangle + \Phi(x)}$$

$$x^* = \int_{\Omega^*} \xi . p_x(\xi) d\xi$$

FOURIER/LAPLACE TRANSFORM

$$\Psi(x) = -\log \Phi(x) = -\log \int_{\Omega^*} e^{-\langle x, y \rangle} dy$$

ENTROPY= LEGENDRE(- LOG[LAPLACE])

Legendre Transform of minus logarithm of characteristic function (Laplace transform) = ENTROPY

THALES

Jean-Marie Souriau

- Graduated from ENS ULM (Ecole Normale Supérieure Paris), with Elie Cartan teacher in 1945

- Souriau PhD at ONERA: J.M. Souriau, “Sur la Stabilité des Avions” ONERA Publ., 62, vi+94, 1953 (proof that you can stabilize one aicraft with respect to all positions of engine: Caravelle)

- Algèbre Multi-Linéaire: J.M. Souriau, Calcul linéaire, P.U.F., Paris, 1964; Algorithme de Le Verrier-Souriau (équation des paramètres du polynôme caractéristique)

$$P(\lambda) = \det(\lambda I - A) = k_0\lambda^n + k_1\lambda^{n-1} + \cdots + k_{n-1}\lambda + k_n$$

$$Q(\lambda) = \mathrm{Adj}(\lambda I - A) = \lambda^{n-1}B_0 + \lambda^{n-2}B_1 + \cdots + \lambda B_{n-2} + B_{n-1}$$

$$k_0 = 1 \quad \text{et} \quad B_0 = I$$

$$A_i = B_{i-1}A, \qquad k_i = -\frac{1}{i}\mathrm{tr}(A_i), \qquad B_i = A_i + k_i I$$

$$A_n = B_{n-1}A \quad \text{et} \quad k_n = -\frac{1}{n}\mathrm{tr}(A_n)$$

« Ce que Lagrange a vu, que n'a pas vu Laplace, c’était la structure symplectique »

- Introduit de la géométrie Symplectique en Mécanique (relecture de Lagrange): J.M. Souriau, Structure des systèmes dynamiques, Dunod, Paris, 1970

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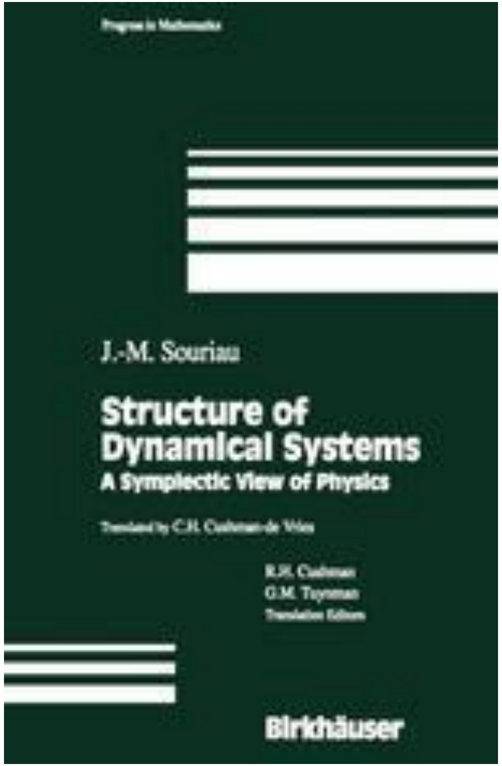
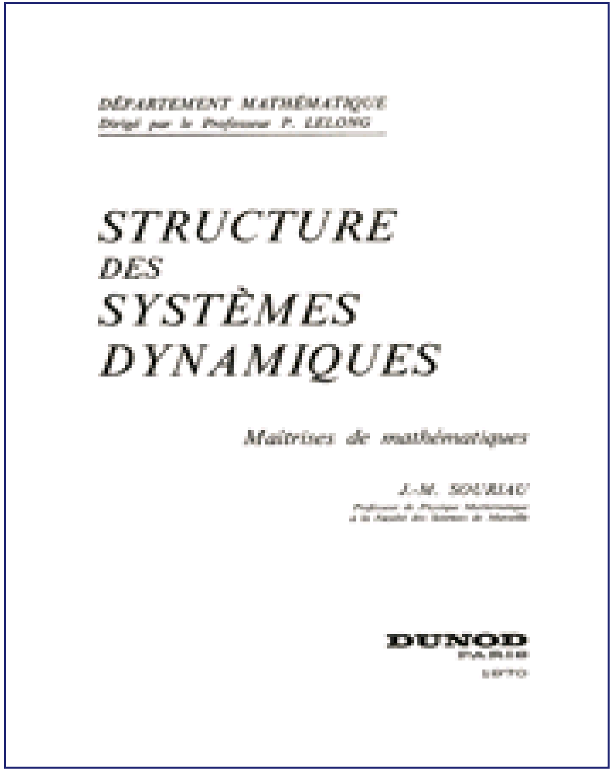
Fundamental Book of Jean-Marie Souriau





http://www.jmsouriau.com/structure_des_systemes_dynamiques.htm
http://www.springer.com/us/book/9780817636951

- Introduction of Symplectic Geometry in Mechanics
- Invention of Moment(um) application
- Geometrization of Noether theorem
- Baycentric Decomposition Theorem
- Total mass of an isolated dynamic system is the class of cohomology of the equivariance default of momentum application.
- Lie Group Thermodynamics (Chapitre IV)

Main references for « Lie Group Thermodynamics »

SUPPLEMENTO AL NUOVO CIMENTO
VOLUME IV

N. 1, 1966

Définition covariante des équilibres thermodynamiques.

J.-M. Souriau
Faculté des Sciences - Marseille

(ricevuto il 5 Novembre 1965)

Contents. — 1. Un problème variationnel. – 2. Mécanique statistique classique. – 3. Equilibres permis par un groupe de Lie. – 4. Exemples. – 5. Localisation de la température vectorielle.

1966

Colloques Internationaux C.N.R.S.
N° 237 – Géométrie symplectique et physique mathématique

MÉCANIQUE STATISTIQUE,
GROUPES DE LIE ET COSMOLOGIE

Jean-Marie SOURIAU (1)

Première partie

FORMULATION SYMPLECTIQUE DE LA MECANIQUE STATISTIQUE

1974

Gibbs Equilibrium is not covariant with respect to Dynamic Groups of Physics

MÉCANIQUE STATISTIQUE COVARIANTE

Le groupe des translations dans le temps (7.9) est un sous-groupe du groupe de Galilée ; mais *ce n'est pas un sous-groupe invariant*, ainsi que le montre un calcul trivial. Si un système dynamique est *conservatif* dans un repère d'inertie, il en résulte qu'il peut *ne plus être conservatif dans un autre*. La formulation (17.24) du principe de Gibbs doit donc être élargie, pour devenir compatible avec la relativité galiléenne.

Nous proposons donc le principe suivant :

(17.77) Si un système dynamique est invariant par un sous-groupe de Lie G' du groupe de Galilée, les équilibres naturels du système constituent l'ensemble de Gibbs du groupe dynamique G'.

Soit $\mathcal{G}'$ l'algèbre de Lie G' ; on sait que $\mathcal{G}'$ est une sous-algèbre de Lie de celle de G, notée $\mathcal{G}$; un équilibre du système sera caractérisé par un élément Z de $\mathcal{G}'$, donc de $\mathcal{G}$; on pourra écrire

(17.78)
$$Z = \begin{bmatrix} j(\omega) & \beta & \gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{bmatrix}$$

en utilisant les notations (13.4) ; Z parcourt l'ensemble Ω défini en (16.219) ; à chaque valeur de Z est associé un élément M du dual $\mathcal{G}'^*$ de $\mathcal{G}'$, valeur moyenne du moment μ ; on peut appliquer les formules (16.219), (16.220), qui généralisent les relations thermodynamiques (17.26), (17.27), (17.28).

(17.79) On voit que c'est Z (17.78) qui généralise la « température » ; le théorème d'isothermie (17.32) s'étend immédiatement : l'équilibre d'un système composé de plusieurs parties sans interactions s'obtient en attribuant à chaque composante un équilibre *correspondant à la même valeur de* Z ; l'entropie s, le potentiel de Planck z et le moment moyen M sont *additifs*. W

J.M. Souriau, Structure des systèmes dynamiques, Chapitre IV « Mécanique Statistique »

Trompette de Souriau

Lorsque le fait qu'on rencontre est en opposition avec une théorie régnante, il faut accepter le fait et abandonner la théorie, alors même que celle-ci, soutenue par de grands noms, est généralement adoptée
- Claude Bernard in "Introduction à l'Étude de la Médecine Expérimentale"

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Example of Gallileo Group

- **The Galileo group of an observer is the group of affine maps**

$$\begin{cases} \vec{x}' = R.\vec{x} + \vec{u}.t + \vec{w} \\ t' = t + e \end{cases}$$

$$\vec{x}, \vec{u} \text{ and } \vec{w} \in R^3, e \in R^+$$

$$R \in SO(3)$$

- **Matrix Form of Gallileo Group**

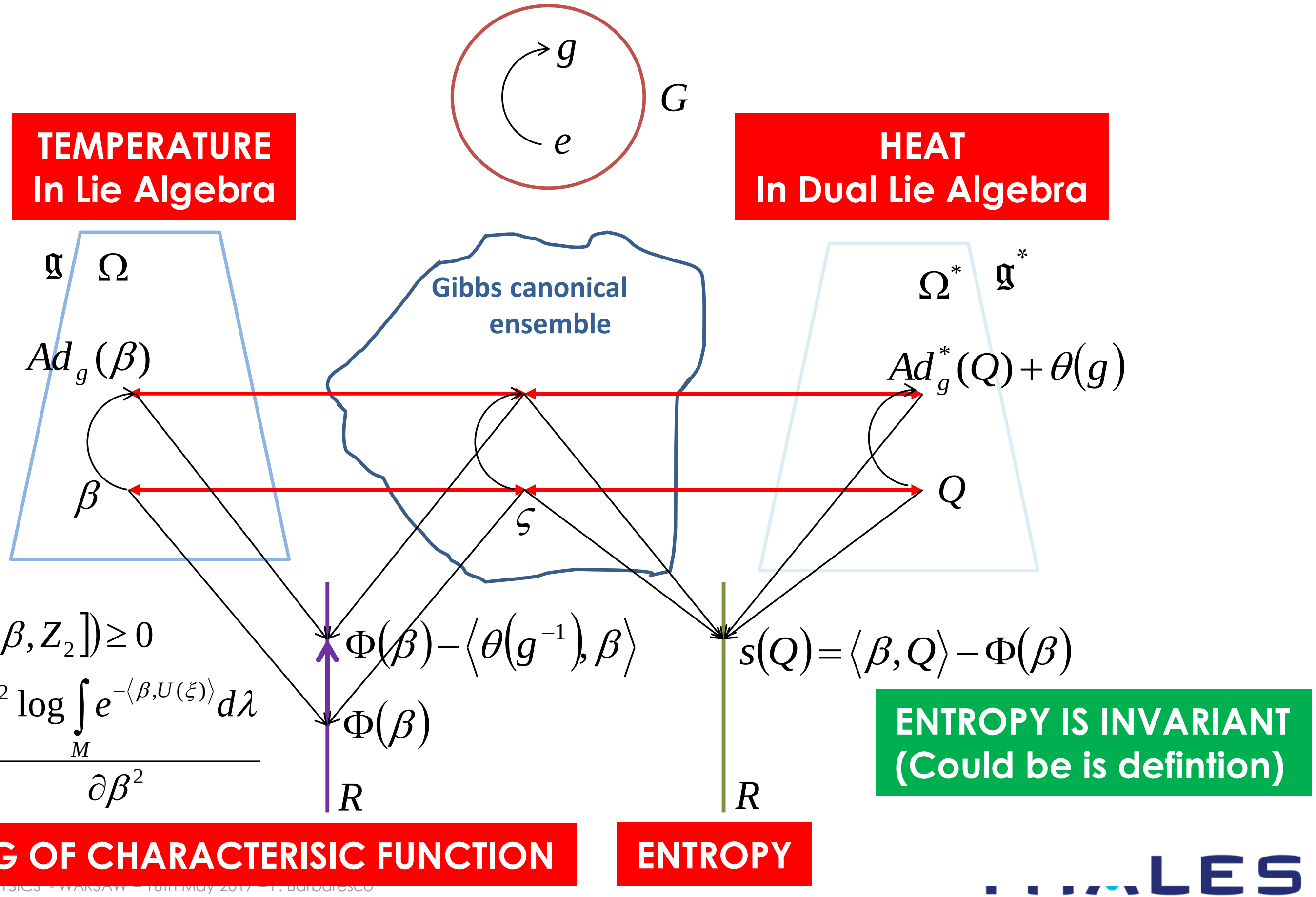
$$\begin{bmatrix} \vec{x}' \\ t' \\ 1 \end{bmatrix} = \begin{bmatrix} R & \vec{u} & \vec{w} \\ 0 & 1 & e \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{x} \\ t \\ 1 \end{bmatrix}$$

- **Symplectic cocycles of the Galilean group: V. Bargmann (Ann. Math. 59, 1954, pp 1–46) has proven that the symplectic cohomology space of the Galilean group is one-dimensional.**

- **Lie Algebra of Gallileo Group**

$$\begin{bmatrix} \vec{\omega} & \vec{\eta} & \vec{\gamma} \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{bmatrix}, \begin{cases} \vec{\eta} \text{ and } \vec{\gamma} \in R^3, \varepsilon \in R^+ \\ \vec{\omega} \in so(3) : \vec{x} \mapsto \vec{\omega} \times \vec{x} \end{cases}$$

Souriau Model of Lie Group Thermodynamics

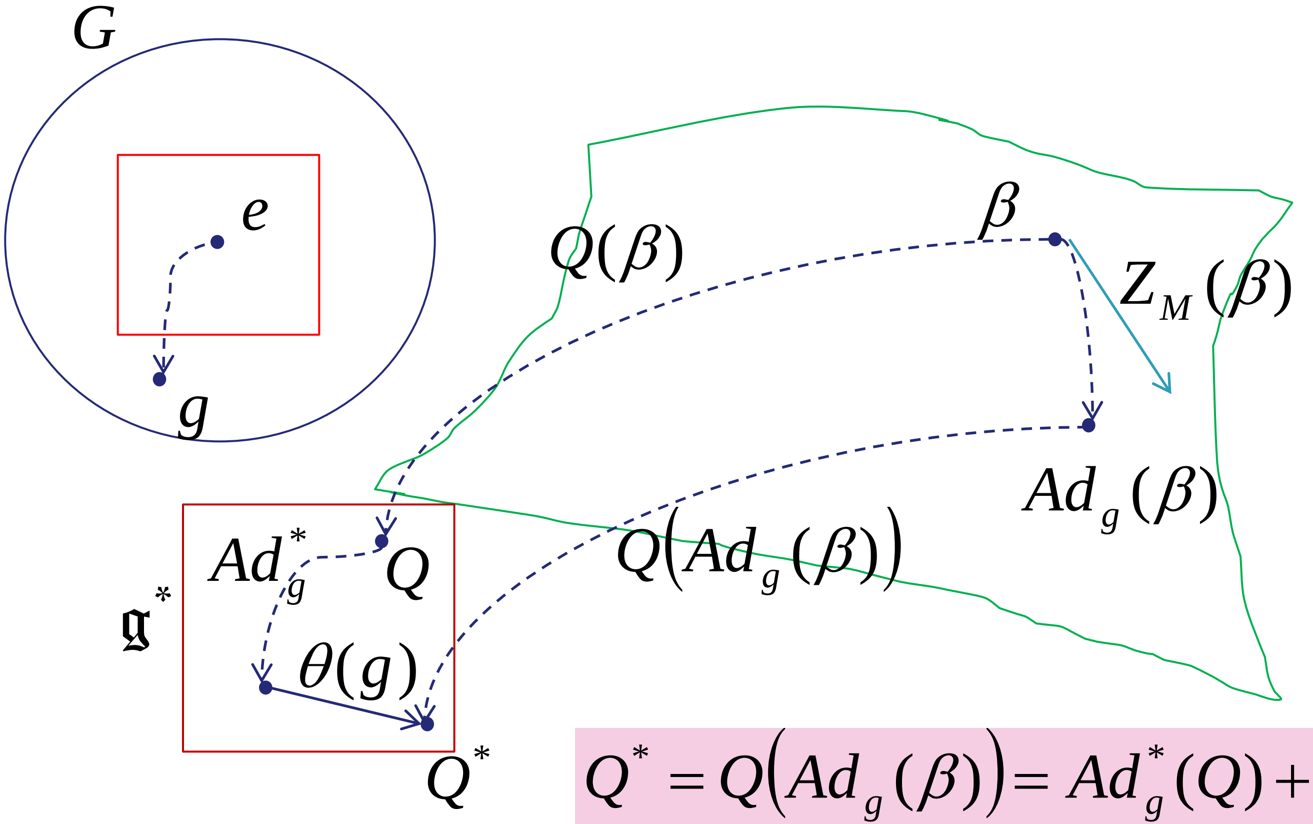


FISHER METRIC

$$g_\beta([\beta,Z_1],[\beta,Z_2])=\tilde{\Theta}_\beta(Z_1,[\beta,Z_2])\geq 0$$

$$I(\beta)=I(Ad_g(\beta))=-\frac{\partial^2\Phi}{\partial\beta^2}=\frac{\partial^2\log\int_M e^{-\langle\beta,U(\xi)\rangle}d\lambda}{\partial\beta^2}$$

Lie Group Action on Symplectic Manifold



$$Q^* = Q\left(Ad_g(\beta)\right) = Ad_g^*(Q) + \theta(g)$$

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High Order Temperature Model by R.S. Ingarden

Ingarden Model compliant with Souriau Model

- We have seen that Souriau has replaced classical Maximum Entropy approach by replacing Lagrange paramete only one geometric "temperature vector" as element of Lie algebra. In parallel, R.S. Ingarden has introduced [232, 233] second and higher order temperature of the Gibbs state that could be extended to Souriau theory of Thermodynamics. Ingarden higher order temperatures could be defined in the case when no variational is considered, but when a probability distribution depending on more than one parameter. R.S. Ingarden has observed that can fail if the following assumptions are not fulfilled: the number of components of the sum goes to infinity and the components of the sum are stochastically independent. Gibbs hypothesis can also fail if stochastic interactions with the environment are not sufficiently weak. In all these cases, we never observe absolute thermal equilibrium of Gibbs type but only flows or turbulence. Nonequilibrium thermodynamics could be indirectly addressed by means of the concept of high order temperatures.

Ingarden references

- Ingarden R.S., Geometry of Thermodynamics, H.D. Doebner and J.D. Hennig (eds), Differential Geometric Methods in Theoretical Physics, pp.455-465, XV DGM Conference, Clausthal,1986
- Ingarden R.S., Gorniewicz L., Shape as an information-thermodynamical concept, and the pattern recognition problem, Math . Nach. 145, 97-109, 1990
- Ingarden, R.S. & Nakagomi T. The Second Order Extension of the Gibbs State, Open Systems & Information Dynamics, vol.1, n°2, pp.243-258, 1992
- Ingarden R.., Towards mesoscopic thermodynamics: Small systems in higher-order states, Open Systems & Information Dynamics archive, Volume 1 Issue 2, June 1992, Pages 309-309
- Ingarden, R.S. & Meller J. Temperature in Linguistics as a Model of Thermodynamics, Open Systems & Information Dynamics vol.2, n°2, pp.211-230, 1994

High Order Temperature Model by R.S. Ingarden

High order thermodynamics

- High order moments: $$Q_k = \frac{\partial \Phi(\beta_1,...,\beta_n)}{\partial \beta_k} = \frac{\int_M U^k(\xi).e^{-\sum_{k=1}^{n}\langle \beta_k , U^k(\xi)\rangle} d\omega}{\int_M e^{-\sum_{k=1}^{n}\langle \beta_k , U^k(\xi)\rangle} d\omega}$$
- High order characteristic function: $$\Phi(\beta_1,...,\beta_n) = -\log \int_M e^{-\sum_{k=1}^{n}\langle \beta_k , U^k(\xi)\rangle} d\omega$$
- High order temperatures and capacities: $$\beta_k = \frac{\partial S(Q_1,...,Q_n)}{\partial Q_k} \qquad K_k = -\frac{\partial Q_k}{\partial \beta_k}$$
- Entropy: $$S(Q_1,...,Q_n) = \sum_{k=1}^{n}\langle \beta_k , Q_k \rangle - \Phi(\beta_1,...,\beta_n)$$
- High order Gibbs density: $$p_{Gibbs}(\xi) = e^{\sum_{k=1}^{n}\langle \beta_k , U^k(\xi)\rangle - \Phi(\beta_1,...,\beta_n)} = \frac{e^{-\sum_{k=1}^{n}\langle \beta_k , U^k(\xi)\rangle}}{\int_M e^{-\sum_{k=1}^{n}\langle \beta_k , U^k(\xi)\rangle} d\omega}$$

Project of 2nd Edition of Souriau Book (send by Claude Vallée)

J. M. SOURIAU

DYNAMIC SYSTEMS STRUCTURE

ENTROPIE

Entropy Definition by Souriau

Lemme :

(17.190) Soit X un espace localement compact; soit λ une mesure positive de X ayant X comme support.

Alors la fonction Φ :

$$\Phi(h) = \text{Log} \int_X e^{h(x)} \lambda(x)\, dx \quad [\forall h \in \mathcal{C}(X) \text{ tel que l'intégrale converge}$$

est convexe.

D'après (17.141), l'intégrale est strictement positive lorsqu'elle est convergente, ce qui assure l'existence de son logarithme. L'épigraphe de Φ (16.46) est l'ensemble des $\binom{h}{y}$ tels que $\int_X e^{h(x)-y} \lambda(x)\, dx \leq 1$; la convexité de l'exponentielle montre que cet épigraphe est convexe.

C.Q.F.D.

Nous allons définir – sous le nom de néguentropie – une transformée de Legendre formelle de cette fonction Φ :

Définition (suite de (17.190))

(17.191) Nous appellerons loi de Boltzmann (relative à λ) toute mesure μ de X telle que l'ensemble de réels

$$\mu(h) - \Phi(h) \quad \left[\begin{array}{l} h \in \text{def}(\Phi) \\ \text{et} \\ h \quad \mu\text{-intégrable} \end{array}\right]$$

TRANSFORMATION DE LAPLACE.

Définition, théorème:

(17.17?) Soit E un espace vectoriel de dimension finie, μ une mesure de son dual E^*. Alors la fonction définie par

$$\Theta \mapsto \int_E e^{M\Theta}\, \mu(M)\, dM \qquad [M \in E^*]$$

pour tout $\Theta \in E$ tel que l'intégrale soit convergente s'appelle transformée de Laplace de μ ; son ensemble de définition est un convexe de E.

p.c. Soient $\Theta_1, \Theta_2 \in \text{def}(F)$, F désignant cette transformée de Laplace; Soit $s \in [0,1]$, $\Theta = [1-s]\Theta_1 + s\,\Theta_2$. Les fonctions $f_1 : M \mapsto e^{M\Theta_1}$, $f_2 : M \mapsto e^{M\Theta_2}$ sont μ-intégrables; il faut montrer que $f : M \mapsto e^{M\Theta}$ est aussi μ-intégrable. Or la fonction $\varphi = [1-s]\, f_1 + s\, f_2$ est μ-intégrable (17.106), donc $|\mu|$-intégrable (17.110); comme $x \mapsto e^x$ est convexe (16.5?) (16.47), on a $e^{M\Theta} = e^{[1-s]\Theta_1 + s\Theta_2} \leq [1-s]\, e^{M\Theta_1} + s\, e^{M\Theta_2}$; donc $|f| \leq \varphi$; f est μ-intégrable (17.109).

C.Q.F.D.

Théorème:

(17.195) Soit E un espace vectoriel de dimension finie, μ une mesure positive non nulle du dual E^*, F sa transformée de Laplace (17.17?). Alors :

1°) F est une fonction convexe semi-continue; $F(\Theta) > 0 \quad \forall \Theta \in \text{def}(F)$;

2°) $f = \text{Log} \circ F$ est convexe et semi-continue ;

3°) Soit Θ un point intérieur de $\text{def}(F)$. Alors :

a) $D^2(f)(\Theta) \geq 0$;

b) $D^2(f)(\Theta) = \int_{E^*} e^{M.\Theta} [M - D(f)(\Theta)]^{\otimes 2} \mu(M)\, dM$

c) $[D^2(f)(\Theta) \text{ inversible}] \Leftrightarrow [\text{enveloppe affine}(\text{support}(\mu)) = E^*]$

Vérification laissée au lecteur (quelques indications: pour montrer le 2°, vérifier que l'épigraphe de f est convexe et fermé; pour vérifier 3°b, considérer les vecteurs V qui font éventuellement partie du noyau de $D^2(f)(\Theta)$)

Generalization: diffeology for groups of diffeomorphisms

GEOMETRIE DES ORBITES COADJOINTES DES GROUPES DE DIFFEOMORPHISMES

Paul Donato

1 - Introduction

On sait l'importance des structures symplectiques dans la description des systèmes dynamiques. On trouve sur les orbites coadjointes d'un groupes de Lie G une structure symplectique à partir de laquelle on peut déterminer toutes les orbites symplectiques de G . Cette structure est utilisée pour la construction de représentations irréductibles des groupes de Lie, et partant, pour la quantification des systèmes. Que subsiste-il de cette géométrie pour les orbites coadjointes de groupes de difféomorphismes d'une variété? J.-M. Souriau a montré que toute variété symplectique (X,σ) préquantifiable (i.e. base d'un fibré principal en cercles $\pi{:}Y\to X$, où Y est muni d'une 1-forme de contact ω telle que $d\omega = \pi^*\sigma$) est une orbite coadjointe d'un groupe de difféomorphismes.

[8] Souriau J.-M. - *Structures des systèmes dynamiques* - Dunod, Paris (1970) .

[9] Souriau J.-M. - *Groupes différentiels* - Lect. Notes in Math. 836 p.81,Springer (1981)

[10] Souriau J.-M.- *Un algorithme générateur de structures quantiques* - Actes du colloque Henri Cartan Lyon, (1984)

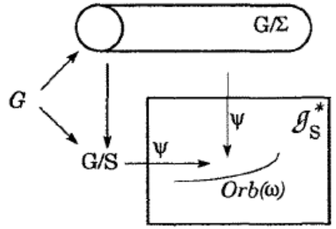
Proposition 5.2 (Souriau [10]) *La forme ω passe au quotient G/Σ en une 1-forme ω_0 invariante par l'action naturelle de G sur son quotient. De plus $d\omega_0$ est l'image réciproque de σ par la projection $G/\Sigma \to G/S$. Il existe une application $\psi{:}G/\Sigma \to \mathfrak{g}_S^*$ vérifiant :*

(i) $\psi(a\Sigma) = [g \longmapsto ga\Sigma]^*\omega_0 = r(a)\omega$

(ii) $\psi(a\Sigma) = \psi(b\Sigma) \Rightarrow aS = bS$

ψ passe à G/S en une application injective appelée application moment *(notée ψ par abus).*

On a le schéma de *préquantification* :



☞ Dans le cas d'une orbite coadjointe préquantifiable d'un groupe de dimension finie l'injectivité du moment est équivalente à la régularité de σ, aussi, dans le cas général, la survivance de l'injectivité peut être considérée comme celle de la régularité de la 2-forme fermée.

Moralité: la structure symplectique des orbites coadjointes des groupes de Lie de dimension finie trouve une généralisation aux orbites préquantifiables des groupes de difféomorphismes.

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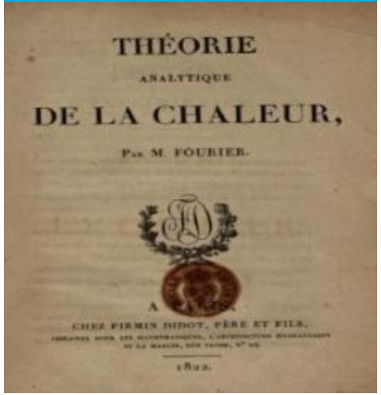
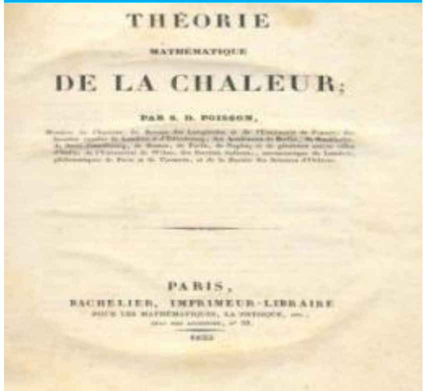
Geometric Theory of Heat

Fourier: Analytical Theory of Heat	Clausius: Mechanical Theory of Heat	Poisson: Mathematical Theory of Heat





Gaston Bachelard Book

- Étude sur l'évolution d'un problème de physique: La propagation thermique dans les solides; Vrin - Bibliothèque des Textes Philosophiques, 1973
- Evolution of ideas : Biot, Fourier, Poisson, Lamé, Boussinesq
- http://www.vrin.fr/book.php?code=9782711600434&search_back=bachelard&editor_back=10

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Reference Book: Libermann & Marle

Symplectic Geometry and Analytical Mechanics

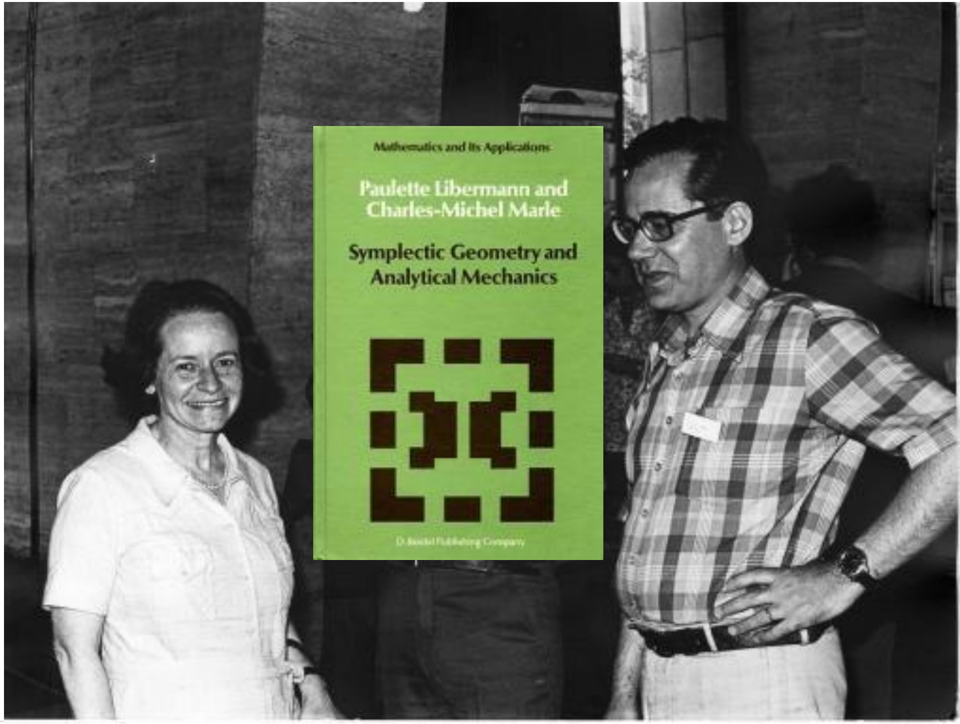
- Paulette Libermann & Charles-Michel Marle

https://www.agnesscott.edu/lriddle/WOMEN/abstracts/libermann_abstract.htm

Paulette Libermann, Legendre foliations on contact manifolds, Differential Geometry and Its Applications, n°1, pp.57-76, 1991

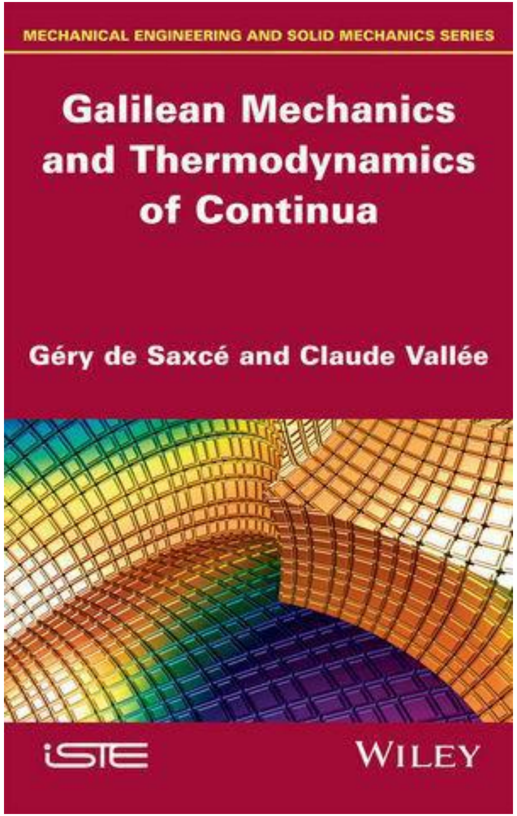
See also:

- Marle, C.-M. From Tools in Symplectic and Poisson Geometry to J.-M. Souriau's Theories of Statistical Mechanics and Thermodynamics. Entropy 2016, 18, 370.
- http://www.mdpi.com/1099-4300/18/10/370



Galilean Mechanics and Thermodynamics of Continua

Book of Géry de Saxcé and Claude Vallée



Galilean Mechanics and Thermodynamics of Continua
Gery de Saxce, Claude Vallee
ISBN: 978-1-84821-642-6
446 pages
February 2016, Wiley-ISTE

http://www.wiley.com/WileyCDA/WileyTitle/productCd-1848216424.html

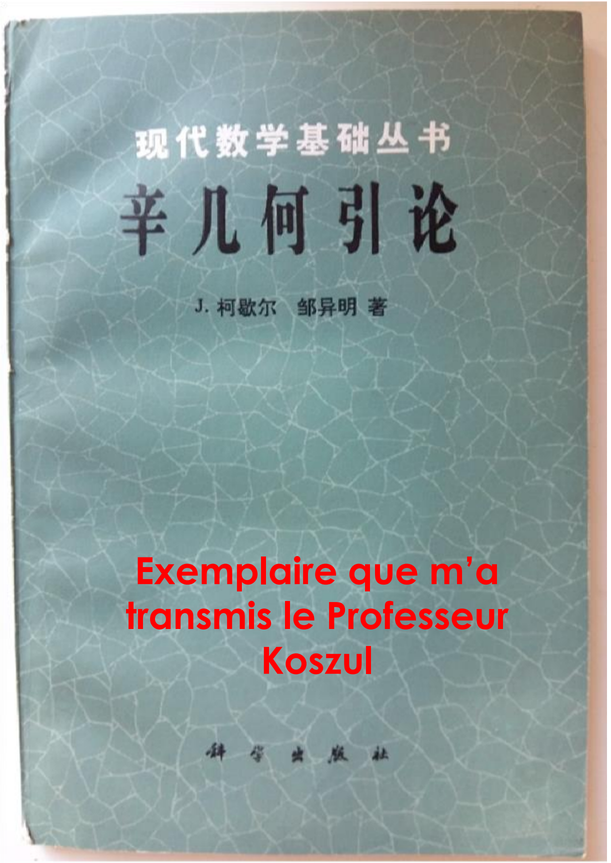
Affine representation of Lie Group and Lie Algebra by Souriau and Koszul

Souriau Model of Affine Representation of Lie Groups and Algebra	Koszul Model of Affine Representation of Lie Groups and Algebra
$A(g)(x) = R(g)(x) + \theta(g)$ with $g \in G, x \in E$ $R : G \to GL(E)$ and $\theta : G \to E$	$Aff(s) : a \mapsto sa = \mathbf{f}(s)a + \mathbf{q}(s) \quad \forall s \in G, \forall a \in E$ $\mathbf{f} : G \to GL(E)$ $s \mapsto \mathbf{f}(s)a = sa - so \quad \forall a \in E$ $\mathbf{q} : G \to E$ $s \mapsto \mathbf{q}(s) = so \quad \forall s \in G$
$\theta(gh) = R(g)(\theta(h)) + \theta(g)$ with $g, h \in G$ $\theta : G \to E$ is a one-cocycle of G with values in E,	$\mathbf{q}(st) = \mathbf{f}(s)\mathbf{q}(t) + \mathbf{q}(s)$
$a(X)(x) = r(X)(x) + \Theta(X)$ with $X \in \mathfrak{g}, x \in E$ The linear map $\Theta : \mathfrak{g} \to E$ is a one-cocycle of G with values in E: $\Theta(X) = T_e\theta\,(X(e)), X \in \mathfrak{g}$	$v \mapsto f(X)v + q(Y)$ f and q the differential of $\mathbf{f}$ and $\mathbf{q}$ respectively
$\Theta\,([X,Y]) = r(X)\,(\Theta(Y)) - r(Y)\,(\Theta(X))$	$q\,([X,Y]) = f(X)q(Y) - f(Y)q(X)\ \forall X, Y \in \mathfrak{g}$ with $f : \mathfrak{g} \to gl(E)$ and $q : \mathfrak{g} \mapsto E$
none	$aff(X) = \begin{bmatrix} f(X) & q(X) \\ 0 & 0 \end{bmatrix}$
none	$Aff(s) = \begin{bmatrix} \mathbf{f}(s) & \mathbf{q}(s) \\ 0 & 1 \end{bmatrix}$

Link between Jean-Louis Koszul and Jean-Marie Souriau: Small Green Book

Jean-Louis Koszul Lecture in China 1986

- "***Introduction à la géométrie symplectique***", in Chinese
- Chuan Yu Ma has written
  - This beautiful, modern book should not be absent from any institutional library. …. During the past eighteen years there has been considerable growth in the research on symplectic geometry. Recent research in this field has been extensive and varied. This work has coincided with developments in the field of analytic mechanics. Many new ideas have also been derived with the help of a great variety of notions from modern algebra, differential geometry, Lie groups, functional analysis, differentiable manifolds and representation theory. [Koszul's book] emphasizes the differential-geometric and topological properties of symplectic manifolds. It gives a modern treatment of the subject that is useful for beginners as well as for experts.



Le chaînon manquant entre Jean-Louis Koszul et Jean-Marie Souriau: Le petit livre vert

On retrouve dans le livre de Koszul, les équations de Souriau:

17.2. 命题. 设 (M, ω) 是一连通的 Hamilton G-空间，

$$\mu: M \to \mathfrak{g}^*$$

是 (M, ω) 的一个矩射，则

(i) 对任意的 $s \in G$，

$$\varphi_\mu(s) = \mu(sx) - Ad^*(s)\mu(x)$$

是 $\mathfrak{g}^*$ 中不依赖于点 $x \in M$ 的一个元素.

(ii) 对任意的 $s, t \in G$ 有

$$\varphi_\mu(st) = \varphi_\mu(s) + Ad^*(s)\varphi_\mu(t).$$

(iii) 对任意的 $a, b \in \mathfrak{g}$ 有

$$c_\mu(a, b) = \langle d\varphi_\mu(a), b \rangle,$$

c_μ 的定义见 §16.

推论. 从 $G \times \mathfrak{g}^*$ 到 $\mathfrak{g}^*$ 内的映射

$$(s, \xi) \longmapsto s\xi = Ad^*(s)\xi + \varphi_\mu(s), \quad s \in G, \xi \in \mathfrak{g}^*,$$

[22] C. L. Siegel, Symplectic geometry, *Amer. J. Math.*, 1—86, 1943.
[23] J. M. Souriau, Structures des systemes dynamiques, Dunod Paris, 1969.
[24] W. W. Symes, Hamiltonian group actions and integrable systems, *Physica*, 1. D, 339—374, 1980.
[25] E. B. Vinberg, The theory of convex homogeneous cones, *Moscow Math. Soc.*, 12, 1963.
[26] N. R. Wallach, Symplectic geometry and Fourier analysis, Math. Sci. Press, Brookline, Mass, 1977.
[27] A. Weil, Variétés kaehleriennes, Hermann, Paris, 1958.
[28] A. Weinstein, Symplectic manifolds and their Lagrangian submanifolds, *Adv. Math.*, 6, 329—346, 1971.
[29] ———, Lectures on symplectic manifolds, C. B. M. S. regional conference series, 29, A. M. S. Rhode Island, 1977.
[30] H. Weyl, Classical groups, Princeton University Press, 1946.
[31] 严志达，半单纯李群李代数表示论，上海科学技术出版社，1963.

THALES

KOSZUL MODEL OF INFORMATION GEOMETRY: Hessian Geometry and Koszul-Vinberg Characteristic Function

OPEN

Fisher Matrix and Fréchet(-Cramer-Rao) Bound

In 1943, Maurice Fréchet published a seminal paper where he introduced first lower bound for all estimator, given by the inverse of Fisher Matrix

- **Fréchet M., Sur l'extension de certaines évaluations statistiques au cas de petits échantillons. Revue de l'Institut International de Statistique 1943, vol. 11, n° 3/4, pp. 182–205.**

$\hat{\theta}$ estimator of θ, Fréchet bound :

$$R_\theta = E\left[\left(\theta-\hat{\theta}\right)\left(\theta-\hat{\theta}\right)^T\right] \geq I(\theta)^{-1}$$

$$I(\theta)\text{ Fisher Matrix : } \left[I(\theta)\right]_{i,j} = -E\left[\frac{\partial^2 \log P(Z/\theta)}{\partial\theta_i\partial\theta_j}\right] \quad \text{with} \quad \theta = \begin{bmatrix}\theta_1\\ \vdots \\ \theta_n\end{bmatrix}$$

- Fréchet informed us that the content of this paper is extracted from his IHP Lecture of Winter 1939 « *Le contenu de ce mémoire a formé une partie de notre cours de statistique mathématique à l'Institut Henri Poincaré pendant l'hiver 1939-1940* ».

Fisher Matrix, Fisher Metric and Information Geometry

- Classically, Fisher Metric of Information Geometry is introduced through Kullback Divergence and 2nd Taylor of ordre 2 expansion:

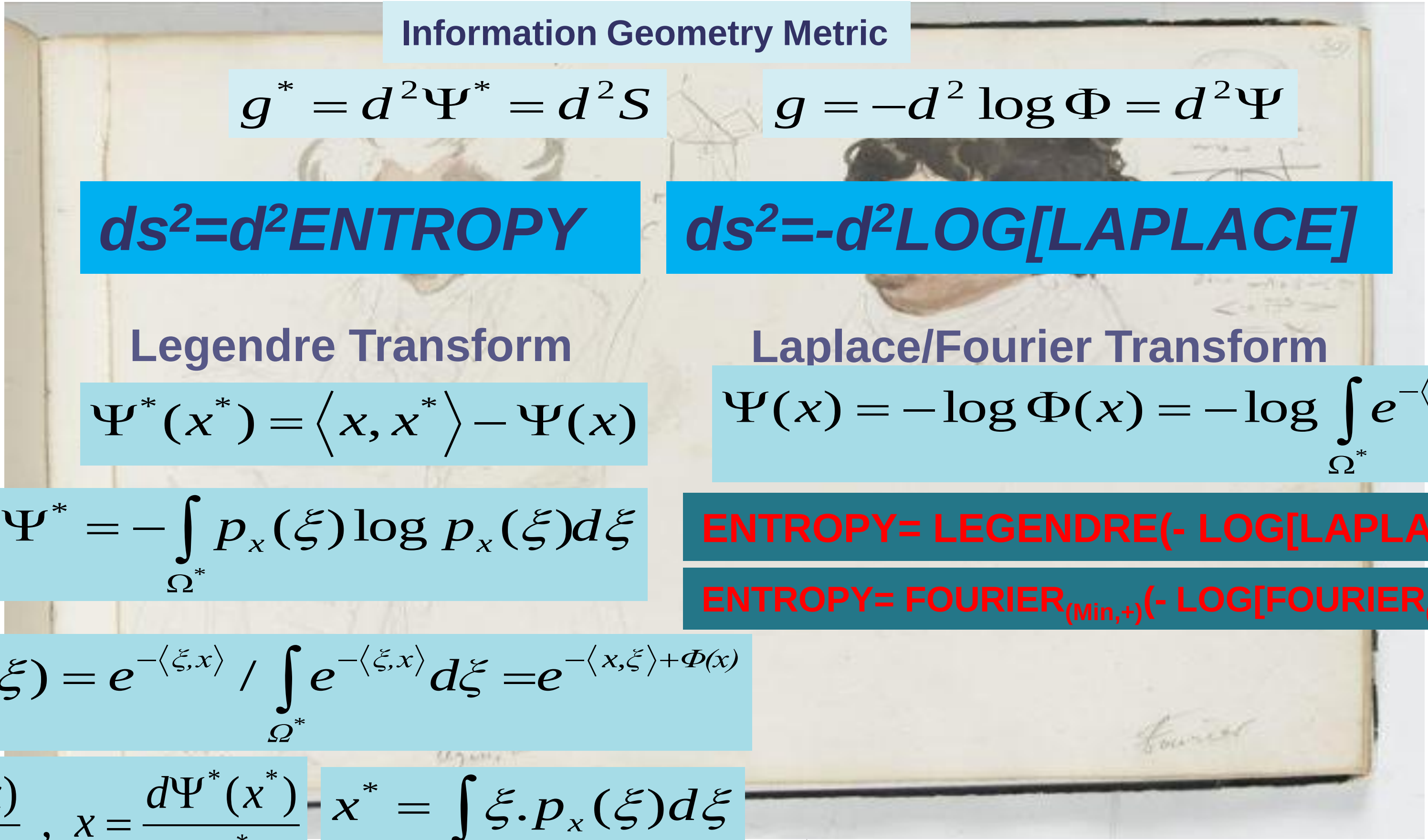
$$ds^2 = K\left[P(Z/\theta), P(Z/\theta + d\theta)\right] \underset{\substack{Taylor\\ ordre\,2}}{=} d\theta^T I(\theta) d\theta = \sum_{i,j=1}^{n} g_{ij} d\theta_i d\theta_j$$

K : Divergence de Kullback

$$K[P,Q] = \int P(Z/\theta)\log\left(\frac{P(Z/\theta)}{Q(Z/\theta)}\right)dZ$$

- Even if we can introduce Kullback divergence (combinatoricstools and stirling formula), this apprach is not fully satisfactory. We prefer to introduce Information Geometry from « **characteristic function** » (introduved by **François Massieu** in Thermodynalics and reused by **Henri Poincaré** in probability) and its developements by **Jean-Louis Koszul**.

Information Geometry and Fisher Metric:
Koszul definition by Koszul(-Vinberg) characateristique function



Information Geometry Metric

$$g^* = d^2\Psi^* = d^2 S \qquad g = -d^2 \log \Phi = d^2 \Psi$$

$ds^2 = d^2$ENTROPY

$ds^2 = -d^2$LOG[LAPLACE]

Legendre Transform

$$\Psi^*(x^*) = \langle x, x^* \rangle - \Psi(x)$$

$$\Psi^* = -\int_{\Omega^*} p_x(\xi) \log p_x(\xi) d\xi$$

Laplace/Fourier Transform

$$\Psi(x) = -\log \Phi(x) = -\log \int_{\Omega^*} e^{-\langle x,y \rangle} dy$$

ENTROPY= LEGENDRE(- LOG[LAPLACE])

ENTROPY= FOURIER$_{(Min,+)}$(- LOG[FOURIER$_{(+,X)}$])

$$p_x(\xi) = e^{-\langle \xi, x \rangle} \Big/ \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi = e^{-\langle x, \xi \rangle + \Phi(x)}$$

$$x^* = \frac{d\Psi(x)}{dx} \;, \; x = \frac{d\Psi^*(x^*)}{dx^*}$$

$$x^* = \int_{\Omega^*} \xi . p_x(\xi) d\xi$$

THALES

Seminal Work of François Jacques Dominique Massieu

Before introducing, Information Geometry with Koszul model, we have to explain the history of « caracteristic function » that was initially introduced in Thermodynamics by (Corps des Mines Engineer) François Jacques Dominique Massieu. See:

- Roger Balian paper from French Academy of Sciences « **François Massieu et les potentiels thermodynamiques** »

http://www.academie-sciences.fr/pdf/hse/evol_Balian2.pdf

- Annale de l'Ecole des Mines: http://www.annales.org/archives/x/massieu.html

1st PhD on « sur les intégrales algébriques (algebraic integrals) » :

- Pour qu'il y ait une intégrale du premier degré dans le mouvement d'un point sur une surface, il faut et il suffit que cette surface soit développable sur une surface de révolution
- Pour qu'il y ait une intégrale du second degré dans le mouvement d'un point sur une surface, il faut et il suffit que cette surface ait son élément linéaire réductible à la forme de Liouville

OPEN

THALES

Development of « Characteristic Function » Concept by Corps des Mines: Massieu, Poincaré, Levy & Balian

1869: François Massieu

- Introduction of Characteristic Function in Thermodynamics
- Use of Massieu Idea by Gibbs and Duhem to define Thermodynamics Potentials

1908-1912: Henri Poincaré (+ Paul Levy)

- Poincaré introduces Characteristic Function in his 1908 Lecture on « Thermodynamics »
- Poincaré introduces Characteristic Function in his 1912 Lecture on « Probability »
- Paul Levy generalizes the Chracteristic function in Probability

1986: Roger Balian

- Balian introduces the Fisher Quantum Metric as hessian of von Neumann Entropy

$$S = \phi - \frac{1}{T}.\frac{\partial \phi}{\partial\left(1/T\right)} = \phi - \beta.U \quad \text{(Tr. Legendre)}$$

S : Entropy, ϕ : Characteristic Function

« Je montre, dans ce mémoire, que toutes les propriétés d'un corps peuvent se déduire d'une fonction unique, que j'appelle la fonction caractéristique de ce corps »
F. Massieu

$$\phi = \log \psi$$

ϕ : *characteristic function (from Massieu)*

ψ : *characteristicfunction (from Poincaré)*

$$\left.\begin{array}{l} S\left(\hat{D}\right) = F\left(\hat{X}\right) - \left\langle \hat{D}, \hat{X} \right\rangle \\ F\left(\hat{X}\right) = \log Tr \exp\left(\hat{X}\right) \end{array}\right\}$$

$$\Rightarrow ds^2 = -d^2 S = Tr\left[d\hat{D} d \log \hat{D}\right]$$

François Massieu: Seminal Idea of Characteristic Function

Main papers of François Massieu

- Massieu, F. **Sur les Fonctions caractéristiques des divers fluides**. Comptes Rendus de l'Académie des Sciences 1869, 69, 858–862.
- Massieu, F. **Addition au précédent Mémoire sur les Fonctions caractéristiques**. Comptes Rendus de l'Académie des Sciences 1869, 69, 1057–1061.
- Massieu, F. **Exposé des principes fondamentaux de la théorie mécanique de la chaleur** (note destinée à servir d'introduction au Mémoire de l'auteur sur les fonctions caractéristiques des divers fluides et la théorie des vapeurs), 31 p., S.l. - s.n., 1873
- Massieu, F. **Thermodynamique: Mémoire sur les Fonctions Caractéristiques des Divers Fluides et sur la Théorie des Vapeurs**; Académie des Sciences: Paris, France, 1876; p. 92.

François Jacques Dominique Massieu : Initial paper on « Characteristic Function » in Rennes

Paper of François Massieu

$$S=\phi-\frac{1}{T}\cdot\frac{\partial\phi}{\partial\left(1/T\right)}=\phi-\beta.U\quad(\text{Tr. Legendre})$$

S : Entropy, ϕ : Characteritic Function

MÉMOIRES

PRÉSENTÉS PAR DIVERS SAVANTS

A L'ACADÉMIE DES SCIENCES

DE L'INSTITUT NATIONAL DE FRANCE.

TOME XXII. — N° 2.

THERMODYNAMIQUE.

MÉMOIRE

SUR

LES FONCTIONS CARACTÉRISTIQUES DES DIVERS FLUIDES

ET SUR LA THÉORIE DES VAPEURS,

PAR F. MASSIEU,

INGÉNIEUR DES MINES, PROFESSEUR À LA FACULTÉ DES SCIENCES DE RENNES.

MÉMOIRES PRÉSENTÉS.

THERMODYNAMIQUE. — *Addition au précédent Mémoire sur les fonctions caractéristiques.* Note de M. F. Massieu, présentée par M. Combes.

» Cette conclusion résultait *à posteriori* de la théorie même; mais j'ai reconnu qu'il était possible de l'établir de prime abord par un procédé qui a l'avantage de conduire plus simplement à la connaissance de la fonction caractéristique et de montrer la liaison de cette fonction avec d'autres fonctions déjà introduites dans la science, savoir : l'entropie S et l'énergie ou chaleur interne U. Je rappellerai d'ailleurs qu'une fois la fonction caractéristique d'un corps déterminée, la théorie thermodynamique de ce corps est faite.

$$\psi=S-\frac{U}{T}.$$

Or, pour avoir S et U, et par suite ψ, il suffit de connaître quelles sont les quantités élémentaires de chaleur dQ qu'il faut fournir au corps suivant un cycle quelconque, pour le faire passer d'un état initial à un état déterminé, et en outre l'accroissement dU de sa chaleur interne pour les différents éléments de ce cycle, ou de tout autre cycle, reliant le même état initial au même état final.

OPEN

THALES

Bad advice of Prof. Joseph Louis François Bertrand to Prof. Massieu

- In following publictions, François Massieu paper is reviewed by Joseph Louis François Bertrand, who give him a **bad advice** to replave variable *1/T* by the variable *T*. If equations seem simpler, Structure support by Legendre transform is broken.

Joseph Louis François Bertrand

(1) Dans le mémoire dont un extrait est inséré aux *Comptes rendus de l'Académie des sciences* du 18 octobre 1869, ainsi que dans la Note additionnelle insérée le 22 novembre suivant, j'avais adopté pour fonction caractéristique $\frac{H}{T}$, ou $S - \frac{U}{T}$; c'est d'après les bons conseils de M. Bertrand que j'y ai substitué la fonction H, dont l'emploi réalise quelques simplifications dans les formules.

- Characteristic Funcion of Massieu and its good parameterization were discovered again by Max Planck (1897) and developed by Herbert Callen (1960) and Roger Balian.

M. Planck

H. Callen

R. Balian

OPEN

Pierre Duhem (1861-1916): Les équations générales de la thermodynamique

Publications de Pierre Duhem:

- Duhem, P. **Sur les équations générales de la thermodynamique**. In Annales scientifiques de l'École Normale Supérieure; Volume 8, pp. 231–266.
- Duhem, P. **Commentaire aux principes de la Thermodynamique—Première partie**. J. Math. Appl. 1892, 8, 269–330.
- Duhem, P. **Commentaire aux principes de la Thermodynamique—Troisième partie**. J. Math. Appl. 1894, 10, 207–286.
- Duhem, P. **Les théories de la chaleur**. Revue des deux Mondes 1895, 130, 851–868.

ihp Institut Henri Poincaré

http://www.duhem2016.info/

UPMC SORBONNE UNIVERSITÉS

Pierre Duhem (1861-1916) et ses contemporains

Institut Henri Poincaré, 14 Septembre 2016

Amphithéâtre Hermite

organisée par Hervé Le Ferrand (Dijon) - Laurent Mazliak (Paris)

Pierre Duhem: General Equations if Thermodynamics

Pierre Duhem and Thermodynamics Potentials

$$\Omega = G(E - TS) + W$$

- Duhem P., « Sur les équations générales de la Thermodynamique », Annales Scientifiques de l'Ecole Normale Supérieure, 3e série, tome VIII, p. 231, 1891

- "Nous avons fait de la Dynamique un cas particulier de la Thermodynamique, une Science qui embrasse dans des principes communs tous les changements d'état des corps, aussi bien les changements de lieu que les changements de qualités physiques"
- « We have done of Dynamics a particular case of Thermodynamics, a Science that could merge in common principles all bodies changes of states, all changes of places as all changes of physical qualities. »

ÉQUATIONS GÉNÉRALES DE LA THERMODYNAMIQUE,

PAR P. DUHEM,

CHARGÉ D'UN COURS COMPLÉMENTAIRE A LA FACULTÉ DES SCIENCES DE LILLE.

Pierre Duhem: General Equations of Thermodynamics

ÉQUATIONS GÉNÉRALES DE LA THERMODYNAMIQUE,

PAR P. DUHEM,

CHARGÉ D'UN COURS COMPLÉMENTAIRE A LA FACULTÉ DES SCIENCES DE LILLE.

Au premier rang, il convient de citer M. F. Massieu ([2]); il a obtenu un résultat capital, à savoir que toutes les équations de la Thermodynamique peuvent être écrites au moyen d'une seule *fonction caractéristique* et de ses dérivées partielles, cette fonction changeant d'ailleurs avec les variables indépendantes adoptées.

M. Gibbs ([3]), dans le Travail célèbre où il a démontré que les fonctions caractéristiques de M. Massieu pouvaient jouer le rôle de potentiels dans la détermination des états d'équilibre du système, a fourni également de profondes idées sur les équations de la Thermodynamique prises sous la forme la plus générale.

M. H. von Helmholtz ([1]) a développé de son côté des idées analogues.

Enfin, M. Arthur von Oettingen ([2]) a donné un exposé de la Thermodynamique d'une remarquable généralité; il a cherché, dans cet exposé, à mettre nettement en évidence le caractère dualistique que présente le développement de la Thermodynamique, caractère déjà marqué par M. Massieu.

([1]) R. CLAUSIUS, *Sur diverses formes des équations fondamentales de la Thermodynamique, qui sont commodes dans l'application* (*Théorie mécanique de la chaleur*. Trad. Folie, Mémoire IX).

([2]) F. MASSIEU, *Sur les fonctions caractéristiques* (*Comptes rendus*, t. LXIX, p. 858 et 1057; 1869). — *Mémoire sur les fonctions caractéristiques des divers fluides et sur la théorie des vapeurs* (*Savants étrangers*, t. XXII; 1876).

([3]) J. WILLARD GIBBS, *On the equilibrium of heterogeneous substances* (*Transactions of the Connecticut Academy*, t. III; 1875-1876).

([1]) H. VON HELMHOLTZ, *Zur Thermodynamik chemischer Vorgänge* (*Sitzungsber. der Berl. Akademie*, t. I, p. 23; 1882).

([2]) ARTHUR VON OETTINGEN, *Die thermodynamischen Beziehungen, antithetisch entwickelt* (*Mémoires de l'Académie de Saint-Pétersbourg*, t. XXIII; 1885).

Duhem developed a general theory and extend the concept of « calorific capacity » (Souriau will gibe a geometrical status to this calorific capacity). Duhem and Souriau theories are compliant

THALES

Pierre Duhem: General Equations of Thermodunamics

> P. DUHEM. Commentaire aux principes de la Thermodynamique (Troisième Partie). Journal de mathématiques pures et appliquées 4e série, tome 10 (1894), p. 207-286.

Le potentiel thermodynamique interne. — Soient

$$U(\alpha, \beta, \ldots, \lambda, \vartheta) \quad \text{et} \quad S(\alpha, \beta, \ldots, \lambda, \vartheta)$$

l'énergie interne et l'entropie d'un système.

Posons

$$(1) \qquad \mathfrak{F}(\alpha, \beta, \ldots, \lambda, \vartheta) = E[U(\alpha, \beta, \ldots, \lambda, \vartheta) - F(\vartheta)S(\alpha, \beta, \ldots, \lambda, \vartheta)].$$

Seconde restriction. — Imaginons qu'à partir d'un certain état $(\alpha, \beta, \ldots, \lambda, \vartheta)$ du système, on lui impose une modification virtuelle

$$\delta\alpha, \quad \delta\beta, \quad \ldots, \quad \delta\lambda, \quad \delta\vartheta.$$

Les actions des corps extérieurs qui le maintiendraient en équilibre en l'état $(\alpha, \beta, \ldots, \lambda, \vartheta)$ effectuent un travail virtuel

$$A\,\delta\alpha + B\,\delta\beta + \ldots + L\,\delta\lambda + \Theta\,\delta\vartheta.$$

$$\begin{cases} A = f_\alpha(\alpha, \beta, \ldots, \lambda, \vartheta), \\ B = f_\beta(\alpha, \beta, \ldots, \lambda, \vartheta), \\ \ldots\ldots\ldots\ldots\ldots\ldots, \\ L = f_\lambda(\alpha, \beta, \ldots, \lambda, \vartheta), \\ \Theta = f_\vartheta(\alpha, \beta, \ldots, \lambda, \vartheta). \end{cases} \rightarrow \begin{cases} R_\alpha = \frac{\partial U}{\partial \alpha} - \frac{f_\alpha}{E}, \\ R_\beta = \frac{\partial U}{\partial \beta} - \frac{f_\beta}{E}, \\ \ldots\ldots\ldots, \\ R_\lambda = \frac{\partial U}{\partial \lambda} - \frac{f_\lambda}{E}, \\ C = \frac{\partial U}{\partial \vartheta} - \frac{f_\vartheta}{E}. \end{cases}$$

$$\begin{cases} R_\alpha = \frac{\partial U}{\partial \alpha} - \frac{A}{E}, \\ R_\beta = \frac{\partial U}{\partial \beta} - \frac{B}{E}, \\ \ldots\ldots\ldots, \\ R_\lambda = \frac{\partial U}{\partial \lambda} - \frac{L}{E}, \\ C = \frac{\partial U}{\partial \vartheta} - \frac{\Theta}{E}. \end{cases} \rightarrow \begin{cases} R_\alpha = F(\vartheta)\frac{\partial S}{\partial \alpha}, \\ R_\beta = F(\vartheta)\frac{\partial S}{\partial \beta}, \\ \ldots\ldots\ldots, \\ R_\lambda = F(\vartheta)\frac{\partial S}{\partial \lambda}, \\ C = F(\vartheta)\frac{\partial S}{\partial \vartheta}. \end{cases} \rightarrow \begin{cases} A = \frac{\partial \mathfrak{F}}{\partial \alpha}, \\ B = \frac{\partial \mathfrak{F}}{\partial \beta}, \\ \ldots\ldots, \\ L = \frac{\partial \mathfrak{F}}{\partial \lambda}, \\ \Theta = \frac{\partial \mathfrak{F}}{\partial \vartheta} + ES\frac{dF(\vartheta)}{d\vartheta}. \end{cases}$$

$$ES = \frac{1}{\frac{dF(\vartheta)}{d\vartheta}}\left(\Theta - \frac{\partial \mathfrak{F}}{\partial \vartheta}\right) \qquad EU = \mathfrak{F} + \frac{F(\vartheta)}{\frac{dF(\vartheta)}{d\vartheta}}\left(\Theta - \frac{\partial \mathfrak{F}}{\partial \vartheta}\right)$$

$$\begin{cases} R_\alpha = \frac{F(\vartheta)}{\frac{dF(\vartheta)}{d\vartheta}}\left(\frac{\partial \Theta}{\partial \alpha} - \frac{\partial^2 \mathfrak{F}}{\partial \vartheta\,\partial \alpha}\right), \\ R_\beta = \frac{F(\vartheta)}{\frac{dF(\vartheta)}{d\vartheta}}\left(\frac{\partial \Theta}{\partial \beta} - \frac{\partial^2 \mathfrak{F}}{\partial \vartheta\,\partial \beta}\right), \\ \ldots\ldots\ldots\ldots\ldots\ldots, \\ R_\lambda = \frac{F(\vartheta)}{\frac{dF(\vartheta)}{d\vartheta}}\left(\frac{\partial \Theta}{\partial \lambda} - \frac{\partial^2 \mathfrak{F}}{\partial \vartheta\,\partial \lambda}\right), \end{cases}$$

$$C = \frac{F(\vartheta)}{\frac{dF(\vartheta)}{d\vartheta}}\left(\frac{\partial \Theta}{\partial \vartheta} - \frac{\partial^2 \mathfrak{F}}{\partial \vartheta^2}\right) - \frac{F(\vartheta)F''(\vartheta)}{[F'(\vartheta)]^2}\left(\Theta - \frac{\partial \mathfrak{F}}{\partial \vartheta}\right)$$

1° *L'expression du potentiel thermodynamique interne du système;*

2° *L'expression*

$$\Theta = f_\vartheta(\alpha, \beta, \ldots, \lambda, \vartheta)$$

de la quantité Θ *relative au système en équilibre,*

On peut déterminer :

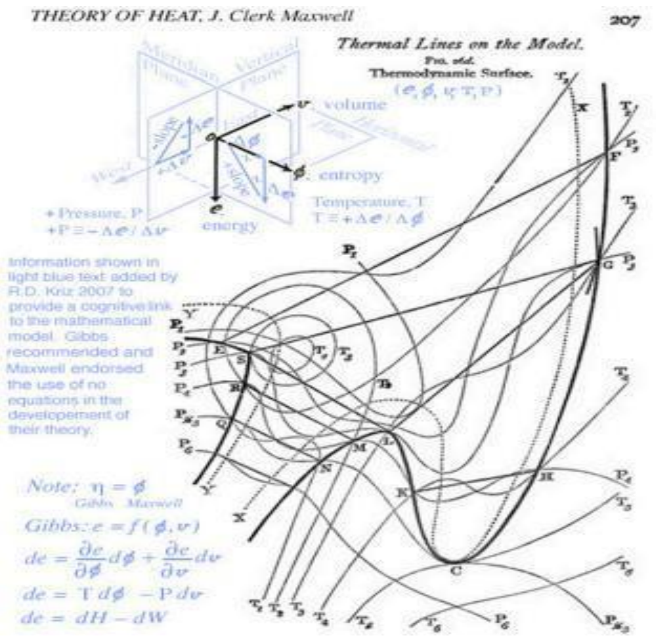
1° *L'énergie interne et l'entropie du système dans un état quelconque;*

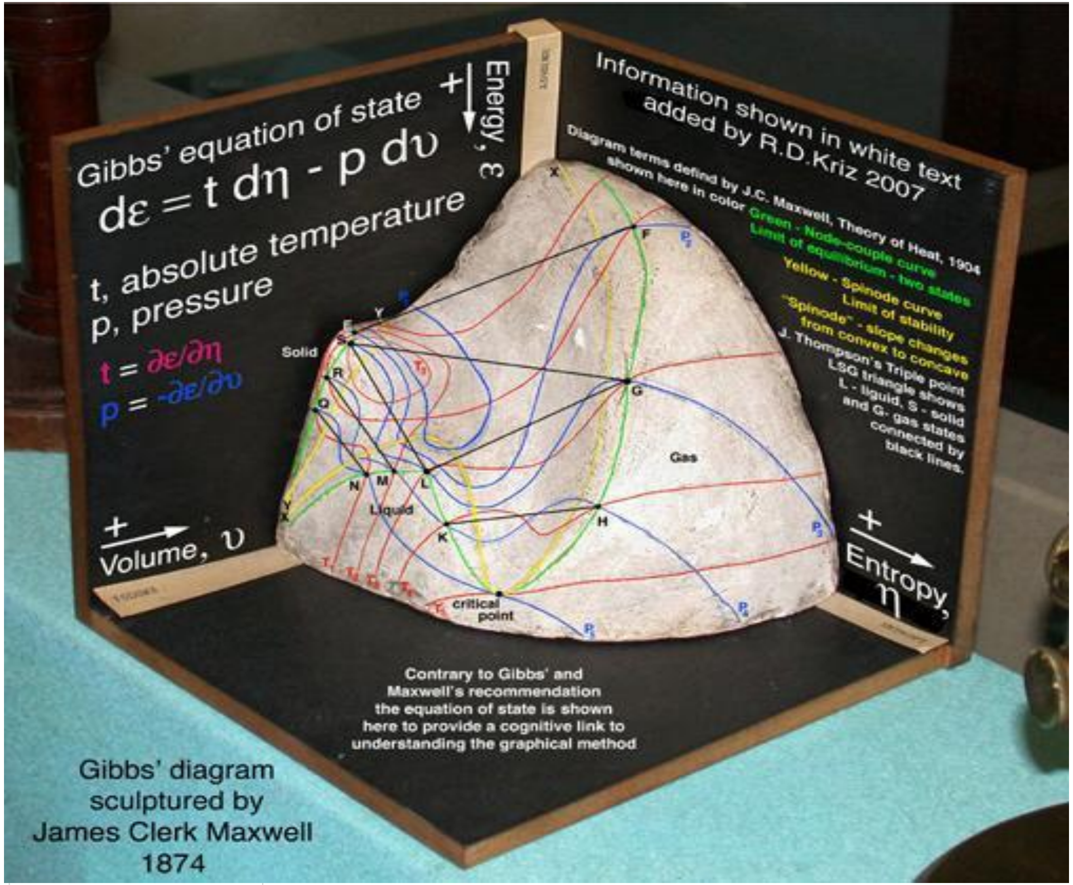
2° *Les conditions nécessaires et suffisantes de l'équilibre du système;*

3° *Les coefficients calorifiques du système en équilibre.*

C'est la généralisation d'une proposition bien connue de M. Massieu.

Gibbs Diagrams sculpted by James Clerk Maxwell (1874)



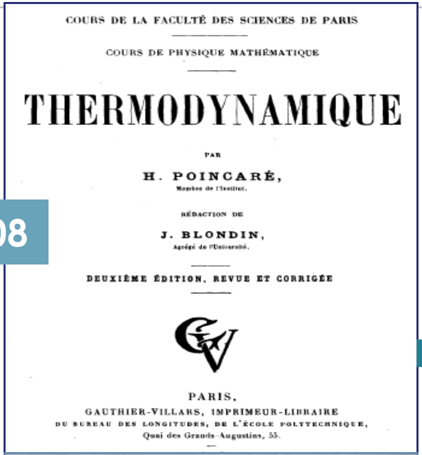


OPEN

THALES

Henri Poincaré Re-Use for Thermodynamics and Probability

1908


COURS DE LA FACULTÉ DES SCIENCES DE PARIS

COURS DE PHYSIQUE MATHÉMATIQUE

THERMODYNAMIQUE

PAR

H. POINCARÉ,

Membre de l'Institut.

RÉDACTION DE

J. BLONDIN,

Agrégé de l'Université.

DEUXIÈME ÉDITION, REVUE ET CORRIGÉE

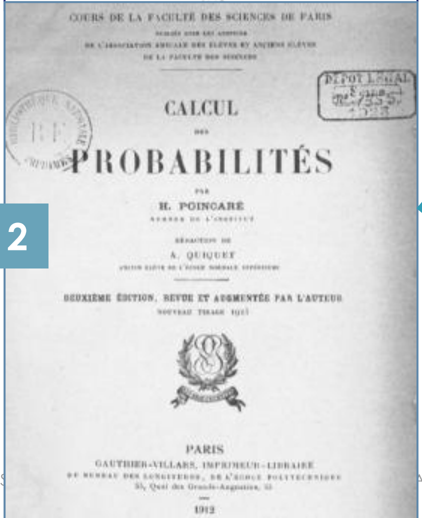
PARIS,

GAUTHIER-VILLARS, IMPRIMEUR-LIBRAIRE

DU BUREAU DES LONGITUDES, DE L'ÉCOLE POLYTECHNIQUE,

Quai des Grands-Augustins, 55.

1912


COURS DE LA FACULTÉ DES SCIENCES DE PARIS

CALCUL

DES

PROBABILITÉS

PAR

H. POINCARÉ

A. QUIQUET

DEUXIÈME ÉDITION, REVUE ET AUGMENTÉE PAR L'AUTEUR

PARIS

GAUTHIER-VILLARS, IMPRIMEUR-LIBRAIRE

1912

125. Fonctions caractéristiques de M. Massieu. — Le théorème de Clausius nous a conduit à l'introduction d'une nouvelle fonction de l'état d'un système : son entropie S.

Si donc nous prenons comme variables indépendantes définissant l'état du système la pression p et le volume spécifique v, nous aurons à considérer, dans les applications, trois fonctions de ces variables : la température T, l'énergie interne U et l'entropie S.

M. Massieu a montré que, si l'on fait choix pour variables indépendantes de v et de T ou de p et de T, il existe une fonction, d'ailleurs inconnue, de laquelle les trois fonctions des variables, p, U et S dans le premier cas, v, U et S dans le second, peuvent se déduire facilement. M. Massieu a donné à cette fonction, dont la forme dépend du choix des variables, le nom de *fonction caractéristique*.

Puisque des fonctions de M. Massieu on peut déduire les autres fonctions des variables, toutes les équations de la Thermodynamique pourront s'écrire de manière à ne plus renfermer que ces fonctions et leurs dérivées ; il en résultera donc, dans certains cas, une grande simplification. Nous verrons bientôt une application importante de ces fonctions.

Henri Poincaré Introduction of Characteristic Function in Probability

Characteristic Function in Probability

- Henri Poincaré introduced « characteristic function » i probability in his Lecture of1912 (inspired by Massieu; both related by logarithm)
- It is ***introduced with Laplace Transform***
- Characteristic function of a real random variable ***defines completly its density of probability***.
- Moments of the random variable could be deduced from successive derivatives at zero of the caracteristic function.
- The 2nd characteristic function is given by the logarithm,: ***generating function of cumulants***.
- **Cumulants** have been introduced in 1889 by danish astronome, mathematician and actuaire **Thorvald Nicolai Thiele** (1838 - 1910). Thiele called them **half-invariants** (demi-invariants).

Fonctions caractéristiques. — J'appelle *fonction caractéristique* $f(\alpha)$ la valeur probable de $e^{\alpha x}$; on aura donc

$$f(\alpha)=\sum p\, e^{\alpha x},$$

si la quantité x varie d'une manière discontinue et peut prendre seulement un nombre fini de valeurs, et

$$f(\alpha)=\int \varphi(x)\, e^{\alpha x}\, dx,$$

si x varie d'une manière continue et si $\varphi(x)$ représente la loi de probabilité. Il est clair que

$$f(\alpha)=1+\frac{\alpha}{1!}(x)+\frac{\alpha^2}{1.2}(x^2)+\frac{\alpha^3}{1.2.3}(x^3)+\ldots,$$

(x^p) désignant la valeur probable de x^p. On voit que $f(0)=1$.

La fonction caractéristique suffit pour définir la loi de probabilité. On a en effet par la formule de Fourier

$$f(i\alpha)=\int_{-\infty}^{+\infty} \varphi(x)\, e^{i\alpha x}\, dx,$$

$$2\pi\varphi(x)=\int_{-\infty}^{+\infty} f(i\alpha)\, e^{-i\alpha x}\, d\alpha.$$

Si deux quantités x et y sont *indépendantes* et si $f(\alpha)$, $f_1(\alpha)$ sont les fonctions caractéristiques correspondantes, la fonction relative à $x+y$ sera le produit $f(\alpha)f_1(\alpha)$. En effet, comme nous l'avons vu au paragraphe 130, la valeur probable du produit $e^{\alpha(x+y)}$ sera le produit des valeurs probables de $e^{\alpha x}$ et $e^{\alpha y}$.

THALES

Quantum Information Geometry of Roger Balian (1/3)

- $Tr\left[\hat{D}\,\hat{O}\right]$ mean values through two dual spaces of observables $\hat{O}$ and of the states $\hat{D}$
- $S = -Tr\left[\hat{D}\log(\hat{D})\right]$ Entropy in space of states
- Entropy S cold be written as a scalar product $S = -\left\langle \hat{D}, \log(\hat{D}) \right\rangle$ where $\log(\hat{D})$ is an element of space of observables, allowing a physical geometrical structure in these spaces.
- The 2nd differential d^2S is a non-negative quadratic form of coordinates of $\hat{D}$ induced by the concavity of the Von Neumann Entropy S . Roger Balian has introduced distance ds btween state $\hat{D}$ and its neiborhood $\hat{D} + d\hat{D}$: $ds^2 = -d^2S = Tr\left[d\hat{D}.d\log\hat{D}\right]$
- Where the Riemannian metric tensor is $-S(\hat{D})$ as function of a set of independant coordinates of $\hat{D}$.

Quantum Information Geometry of Roger Balian (2/3)

- It is possible to introduce the logarithm of a quantum characteristic function $F(\hat{X})$:

$$F(\hat{X}) = \log Tr \exp \hat{X}$$

- Von Neumann Entropy S appears as Legendre transorm of $F(\hat{X})$:

$$S(\hat{D}) = F(\hat{X}) - \langle \hat{D}, \hat{X} \rangle$$

- with $S(\hat{D}) = -Tr\hat{D} \log \hat{D} = -\langle \hat{D}, \log \hat{D} \rangle$

- Where $\hat{X}$ and $\hat{D}$ are conjugate variable of the Legendre transform, making appear the algebraic/geometric duality between $\hat{D}$ and $\log \hat{D}$.

- $F(\hat{X})$ characterizes canonical Thermodynamical equilibrium states with $\hat{X} = \beta.\hat{H}$ and where hamiltonian is $\hat{H}$.

THALES

Quantum Information Geometry of Roger Balian (3/3)

- $dF = Tr\hat{D}d\hat{X}$ with Maximum Entropy Gibbs Density: $\hat{D} = \dfrac{\exp \hat{X}}{Tr\exp \hat{X}}$

- dF are partial derivative of $F(\hat{X})$ with respect to coordinates of $\hat{X}$. $\hat{D}$ Is hermitian, normalised and positifve and can be interpreted as a density matrix .

- Legendre Transform appears with the following development:

$$S(\hat{D}) = -Tr\hat{D}\log\hat{D} = -Tr\left(\hat{D}\left(\hat{X} - \log Tr\exp\hat{X}\right)\right) = -Tr\hat{D}\hat{X} + Tr\left(\hat{D}\right)\log Tr\exp\hat{X}$$

$$Tr\left(\hat{D}\right) = 1 \Rightarrow S(\hat{D}) = F\left(\hat{X}\right) - \left\langle \hat{D}, \hat{X}\right\rangle$$

- Roger Balian has defined the dual Riemanian metric from F, $ds^2 = d^2F$ in conjugate space $\hat{X}$:

$$ds^2 = -dS^2 = Trd\hat{D}d\hat{X} = d^2F$$

- Normalisation of $\hat{D}$ implies $Trd\hat{D} = 0$ and $Trd^2\hat{D} = 0$

Fundamental strcture of Roger Balian Quantum Information Geometry

Legendre Transform

$$S(\hat{D}) = F(\hat{X}) - \left\langle \hat{D}, \hat{X} \right\rangle$$

Von Neumann Entropy

$$S = -Tr\left[\hat{D}\log(\hat{D})\right]$$

Characteristic Function

$$F(\hat{X}) = \log Tr \exp \hat{X}$$

Maximum Entropy Density

$$\hat{D} = \frac{\exp \hat{X}}{Tr \exp \hat{X}}$$

Balian Metric of Quantum Information Geometry (1986)

$$ds^2 = -d^2 S = Tr\left[d\hat{D}.d \log \hat{D}\right]$$

Basic Tool: Duality and Legendre Transform

Legendre Transform plays a central role related to duality & convexity

- Dual Potential Functions (entropy and characteristic function)
- Systems of dual coordinates

Roots of Legendre Transform

- Legendre Transform and Plücker Geometry
- Adrien-Marie Legendre and Gaspard Monge solve Minimal surface problem by use of Legendre Transform
- Chasles and Darboux interpreted Legendre Transform as reciprocal polar with respect to a paraboloid (re-use by Hadamard and Fréchet in calculus of variations)
- Alexis Clairaut introduced previoulsy Clairaut Equation
- Maurice Fréchet introduced Clairaut equation associated to « dstinguished densities » (densities with parameters achieving the Fréchet-Cramer-Rao Bound)

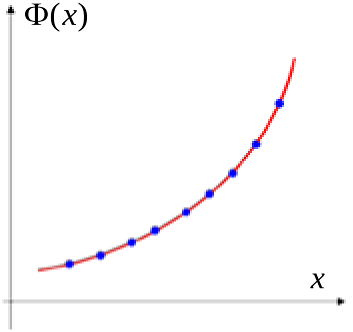
Legendre Transform interpretation

Legendre Transform

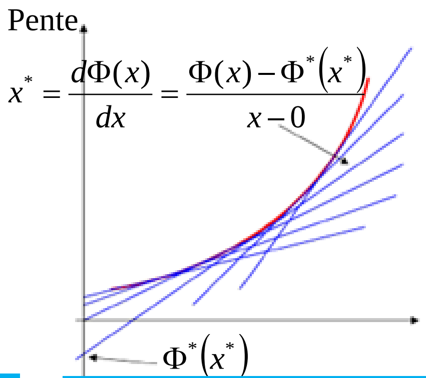
- Legendre Transform transformes one fonction defined by its value in one point in a fonction defined by its tangent.
- Used in thermodynamics and Lagrangian Mechanics

$$\Phi^*(x^*) = \langle x^*, x\rangle - \Phi(x)$$

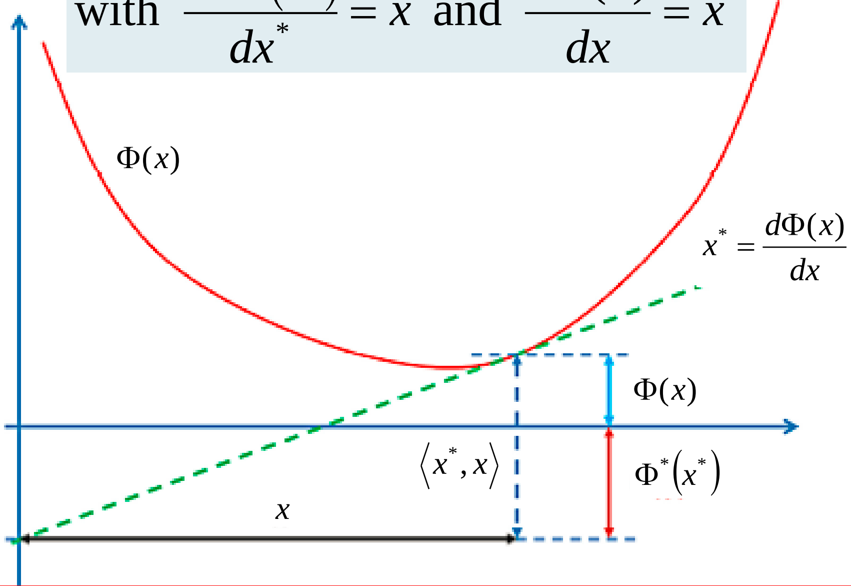
$$\text{with } \frac{d\Phi^*(x^*)}{dx^*} = x \text{ and } \frac{d\Phi(x)}{dx} = x^*$$



Classical Geometry
(curve is given by a continuum of points)



Plücker Geometry
(curve is given bu the envelop of its tangents)



Legendre Transform is equivalent to Fourier Transform for convex function (duality)

Brenier, Yann. Un algorithme rapide pour le calcul de transformées de Legendre-Fenchel discrètes, C. R. Acad. Sci. Paris Sér. I Math. 308 (1989), no. 20, 587–589.

Legendre Transform, 1787

1787, Adrien-Marie Legendre, "Mémoire sur l'intégration de quelques équations aux différences partielles".

- Adrien-Marie Legendre has introduced Lgendre transform to solve a minimal surface problem given by Monge (Monge requested him to consolidate its proof).
- Legendre said "***J'y suis parvenu simplement par un changement de variables qui peut être utile dans d'autres occasions***".

Legendre, A.M. Mémoire Sur L'intégration de Quelques Equations aux Différences Partielles; Mémoires de l'Académie des Sciences: Paris, France, 1787; pp. 309–351.

DES SCIENCES. 309

MÉMOIRE
SUR
L'INTÉGRATION DE QUELQUES ÉQUATIONS AUX DIFFÉRENCES PARTIELLES.

Par M. LE GENDRE.

(I.)

De l'Équation de la moindre Surface.

ON fait, d'après M. de la Grange, que la furface la moindre entre des limites données, a pour équation différentielle

$$(1 + q^2)\frac{ddz}{dx^2} - 2pq\frac{ddz}{dx\,dy} + (1 + p^2)\frac{ddz}{dy^2} = 0,$$

en faifant, pour abréger, $\frac{dz}{dx} = p$, $\frac{dz}{dy} = q$. M. Monge a tenté d'intégrer cette équation dans les Mémoires de l'Académie de 1784; mais l'intégrale qu'il a donnée *(page 149)* n'étant pas à l'abri de toute objection, attendu que les fignes

Lorfque la valeur de ω fera connue, il eft clair qu'on aura celles de x, y, z, exprimées en p & q; favoir,

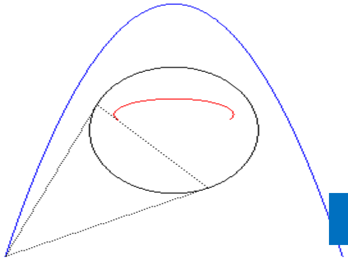
$$x = \frac{d\omega}{dp},\quad y = \frac{d\omega}{dq},$$

$$z = px + qy - \omega.$$

Reciprocal Polar with respect to a paraboloid

Legendre Transform & Reciprocal Polar

- **Darboux** gave in his book one interpretation of **Chasles** : « ***Ce qui revient suivant une remarque de M. Chasles, à substituer à la surface sa polaire réciproque par rapport à un paraboloïde*** »
- In the lecture « Leçons sur le calcul des variations », **J. Hadamard**, followed by **M.E. Vessiot**, used reciprocal polar of figurative, and figuratrice.
- Note of **Paul Belgodère** presented by Elie Cartan « ***Extrémale d'une surface*** »



SUR LA THÉORIE DES MULTIPLICITÉS ET LE CALCUL DES VARIATIONS

PAR M. E. VESSIOT.

COURS DU COLLÈGE DE FRANCE

LEÇONS SUR LE CALCUL DES VARIATIONS

PROFESSÉES PAR J. HADAMARD, PROFESSEUR AU COLLÈGE DE FRANCE

Recueillies par M. FRÉCHET

TOME PREMIER

Partons d'abord du problème de Lagrange, et posons comme précédemment

(200) $q_i = f_{y'_i}, \quad (i = 1, \ldots n)$

(201) $H = \sum_{i=1}^{n} y'_i f_{y'_i} - f.$

La transformation de Legendre, définie par les équations (200), (201) reviendra encore à prendre la polaire réciproque de cette figurative par rapport au paraboloïde

(205) $y'^2_1 + y'^2_2 - 2f_0 = 0.$

On voit bien ainsi comment, de la figurative qui est une courbe, nous déduirons une surface, celle qui est représentée par l'équation (202). Cette surface, — qui est encore dite la *figuratrice* —

Clairaut Equation, 1734

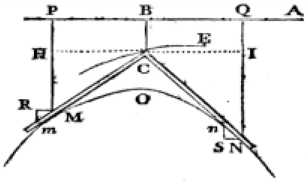
1724, Alexis Claude Clairaut

- Clairaut introduced before Legendre an equation related to Legendre transform.
- Legendre transform

$$\Phi^*(x^*) = \langle x^*, x\rangle - \Phi(x) \quad \text{avec} \quad \frac{d\Phi^*(x^*)}{dx^*} = x \quad \text{et} \quad \frac{d\Phi(x)}{dx} = x^*$$

- **Clairaut Equation**

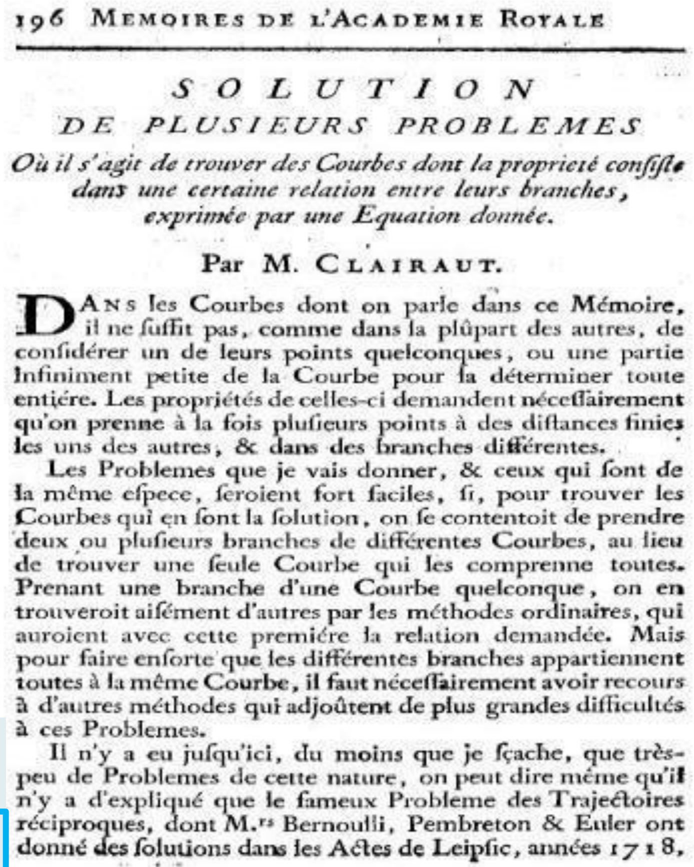
$$\Phi^*(x^*) = \left\langle x^*, \frac{d\Phi^*(x^*)}{dx^*}\right\rangle - \Phi\left(\frac{d\Phi^*(x^*)}{dx^*}\right)$$



- Singular solution: envelolp of straith lines solutions

$$\frac{d\Phi^*(x^*)}{dx^*} = \frac{d\Phi^*(x^*)}{dx^*} + \left\langle x^*, \frac{d^2\Phi^*(x^*)}{dx^{*2}}\right\rangle - \Phi'\left(\frac{d\Phi^*(x^*)}{dx^*}\right)\frac{d^2\Phi^*(x^*)}{dx^{*2}}$$

$$\Rightarrow 0 = \left\langle x^* - \Phi'\left(\frac{d\Phi^*(x^*)}{dx^*}\right), \frac{d^2\Phi^*(x^*)}{dx^{*2}}\right\rangle \Rightarrow x^* = \Phi'\left(\frac{d\Phi^*(x^*)}{dx^*}\right) \text{ et } \Phi(x^*) = \langle C, x^*\rangle + \Phi(C)$$



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SOLUTION
DE PLUSIEURS PROBLEMES

Où il s'agit de trouver des Courbes dont la proprieté conſiſte dans une certaine relation entre leurs branches, exprimée par une Equation donnée.

Par M. CLAIRAUT.

DANS les Courbes dont on parle dans ce Mémoire, il ne ſuffit pas, comme dans la plûpart des autres, de conſidérer un de leurs points quelconques, ou une partie Infiniment petite de la Courbe pour la déterminer toute entiére. Les propriétés de celles-ci demandent néceſſairement qu'on prenne à la fois pluſieurs points à des diſtances finies les uns des autres, & dans des branches différentes.

Les Problemes que je vais donner, & ceux qui ſont de la même eſpece, ſeroient fort faciles, ſi, pour trouver les Courbes qui en ſont la ſolution, on ſe contentoit de prendre deux ou pluſieurs branches de différentes Courbes, au lieu de trouver une ſeule Courbe qui les comprenne toutes. Prenant une branche d'une Courbe quelconque, on en trouveroit aiſément d'autres par les méthodes ordinaires, qui auroient avec cette premiére la relation demandée. Mais pour faire enſorte que les différentes branches appartiennent toutes à la même Courbe, il faut néceſſairement avoir recours à d'autres méthodes qui adjoûtent de plus grandes difficultés à ces Problemes.

Il n'y a eu juſqu'ici, du moins que je ſçache, que très-peu de Problemes de cette nature, on peut dire même qu'il n'y a d'expliqué que le fameux Probleme des Trajectoires réciproques, dont M.rs Bernoulli, Pembreton & Euler ont donné des ſolutions dans les Actes de Leipſic, années 1718,

THALES

Maurice Fréchet et l'équation de Clairaut

Travaux précurseurs de Maurice Fréchet

- En 1939, dans son cours de l'IHP, Maurice Fréchet introduit ce qui fut appelée ensuite borne de Cramer-Rao

$$(\sigma_T)^2 \geqslant \frac{1}{n(\sigma_A)^2} \text{ avec } T = H(X_1, \ldots X_n), \quad A = \frac{1}{f(X,\theta)} \frac{\partial f(X,\theta)}{\partial \theta}$$

$\hat{\theta}$ estimateur de θ, borne de Fréchet : $R_\theta = E\left[\left(\theta - \hat{\theta}\right)\left(\theta - \hat{\theta}\right)^T\right] \geq I(\theta)^{-1}$

$$[I(\theta)]_{i,j} = -E\left[\frac{\partial^2 \log P(Z/\theta)}{\partial\theta_i \partial\theta_j}\right] = E\left[\frac{\partial \log P(Z/\theta)}{\partial\theta_i}\frac{\partial \log P(Z/\theta)}{\partial\theta_j}\right]$$

- Dans son article de 1943, Maurice Fréchet s'intéresse aux "**densités distinguées**", densités qui atteignent cette borne. Il montre qu'elles dépendent d'une fonction (logarithme de la fonction caractéristique) qui vérifie **l'équation de Alexis Clairaut**.

$$\Phi^*(x^*) = \left\langle x^*, \frac{d\Phi^*(x^*)}{dx^*}\right\rangle - \Phi\left(\frac{d\Phi^*(x^*)}{dx^*}\right) \text{ et } x^* = \frac{d\Phi(x)}{dx}$$

(55) $$\mu = \theta\,\mu' - \psi(\mu')$$

c'est-à-dire une équation de Clairaut. La solution $\mu' =$ constante réduirait $f(x, \theta)$, d'après (48) à une fonction indépendante de θ, cas où le problème n'aurait plus de sens. μ est donc donné par la solution singulière de (55), qui est unique et s'obtient en éliminant s entre $\mu = \theta\, s - \psi(s)$ et $\theta = \psi'(s)$ ou encore entre

THALES

Seminal Maurice Fréchet paper 1943

Fréchet, M. Sur l’extension de certaines évaluations statistiques au cas de petits échantillons. Revue de l’Institut International de Statistique 1943, 11, 182–205.

Etude des densités distinguées. Appelons (provisoirement, dans ce mémoire) *densité distinguée*, toute densité de probabilité $f(x, \theta)$ telle que la fonction

(46)
$$\theta + \frac{\dfrac{\partial L f(x,\theta)}{\partial \theta}}{\displaystyle\int_{-\infty}^{+\infty}\left[\frac{\partial}{\partial \theta} f(x,\theta)\right]^2 \frac{dx}{f(x,\theta)}}$$

soit indépendante de θ.

Pour ces densités distinguées, on va pouvoir déterminer la fonction minimisante $H'(X_1, \ldots . X_n)$ et étendre au cas des petits échantillons la comparaison des hodes d’estimation faites par divers auteurs dans le cas des grands échantillons. Il vaut donc la peine de chercher la forme générale de $f(x, \theta)$ pour cette catégorie de variables.

de θ. En appelant $h(x)$ cette fonction, on voit qu’on a l’identité de la forme

(47)
$$\lambda(\theta)\frac{\partial}{\partial \theta} L f(x,\theta) = h(x) - \theta$$

où $\lambda(\theta) > 0$. On peut considérer $\frac{1}{\lambda(\theta)}$ comme la dérivée seconde d’une fonction $\mu(\theta)$; d’où $\frac{\partial}{\partial \theta} L f(x,\theta) = \mu_\Theta''(\theta)\,[h(x) - \theta]$.

Par suite $L f(x,\theta) - \mu'_\Theta\,[h(x) - \theta] - \mu(\theta)$ est une quantité indépendante de θ que nous pouvons représenter par $l(x)$.

Ainsi toute densité distinguée, $f(x, \theta)$, est de la forme

(48)
$$f(x,\theta) = e^{\mu'_\Theta [h(x) - \Theta] + \mu(\Theta) + l(x)}$$

(52bis)
$$\lambda\,\mu'' = 1.$$

Incidemment, puisque, d’après (52), $\lambda(\theta)$ est positif, il en résulte aussi que $\mu''\left(= \frac{1}{\lambda(\theta)}\right)$ est aussi positif.

Fisher metric

On peut d’ailleurs préciser d’une manière plus directe que par (50), le choix des fonctions $\mu(\theta)$, $h(x)$, $l(x)$: *on peut prendre arbitrairement $h(x)$ et $l(x)$* [1] *et alors $\mu(\theta)$ est déterminé par* (50) ou même mieux *par une formule explicite.* En effet, (50) peut s’écrire

$$e^{\Theta\mu' - \mu} = \int_{-\infty}^{+\infty} e^{\mu'\Theta h(x) + l(x)}\,dx.$$

Donnons-nous alors arbitrairement $h(x)$ et $l(x)$ et soit s une variable arbitraire : la fonction

$$\int_{-\infty}^{+\infty} e^{s h(x) + l(x)}\,dx \qquad {}^{1})$$

sera une fonction positive connue que nous pourrons représenter par $e^{\Psi(s)}$. On voit alors que $\mu(\theta)$ sera défini par

$$\theta\,\mu' - \mu = \psi(\mu') \quad \text{ou}$$

(55)
$$\mu = \theta\,\mu' - \psi(\mu')$$

Legendre-Clairaut

c’est-à-dire une équation de Clairaut. La solution μ' = constante réduirait $f(x, \theta)$, d’après (48) à une fonction indépendante de θ, cas où le problème n’aurait plus de sens. μ est donc donné par la solution singulière de (55), qui est unique et s’obtient en éliminant s entre $\mu = \theta\, s - \psi(s)$ et $\theta = \psi'(s)$ ou encore entre

$$e^{\Theta s - \mu} = \int_{-\infty}^{+\infty} e^{s h(x) + l(x)}\,dx \text{ et}$$

(55bis)
$$\int_{-\infty}^{+\infty} e^{s h(x) + l(x)}\,[h(x) - \theta]\,dx = 0.$$

Si l’on veut, $\mu(\theta)$ est donné par la relation

$$e^{-\mu} = e^{-\Theta s}\int_{-\infty}^{+\infty} e^{s h(x) + l(x)}\,dx$$

où s est donné en fonction de θ par la relation implicite (55bis).

Fondation of Information Geometry by Jean-Louis Koszul

Fisher Metric = Hassian Metric

- Incas of Maximum Entropy Density (Gibbs density), Riemannian metric associated to hessian of logarithm of the partition function is equal to Fisher Metric.
- These Hessian Geometrical Structures have been studied in parallel par the mathemacian **Jean-Louis Koszul** and his PhD student **Jacques Vey** in more general framework of sharp convex cones:
  - Koszul Forms
  - Koszul-Vinberg Characteristic Function
- In Jean-Louis Koszul model , fondamental structures are deduced of affine represnetation of Liegroup and Lie Algebra (this affine representation is also at the heat of Jean-Marie Souriau model).

ENS Promo 40

THALES

Jean-Louis Koszul and Hirohiko Shima




Jean-Louis Koszul, Correspondant the French Academy of Sciences, PhD student of Henri Cartan, Bourbaki member

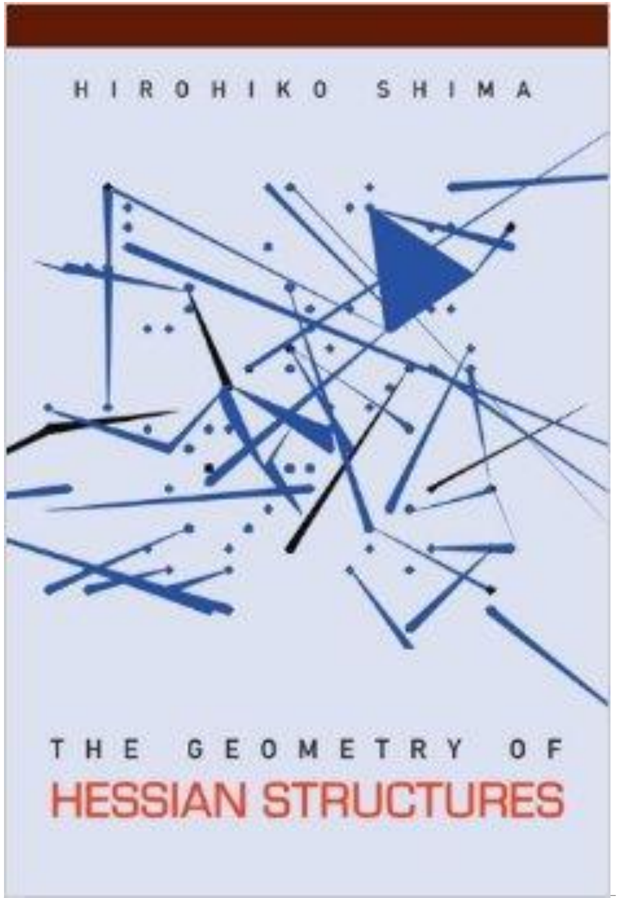
- Koszul, J.L. Sur la forme hermitienne canonique des espaces homogènes complexes. Can. J. Math. 1955, 7, 562–576.
- Koszul, J.L. Exposés sur les Espaces Homogènes Symétriques; Publicação da Sociedade de Matematica de São Paulo: São Paulo, Brazil, 1959.
- Koszul, J.L. Domaines bornées homogènes et orbites de groupes de transformations affines. Bull. Soc. Math. Fr. 1961, 89, 515–533.
- Koszul, J.L. Ouverts convexes homogènes des espaces affines. Math. Z. 1962, 79, 254–259.
- Koszul, J.L. Variétés localement plates et convexité. Osaka. J. Math. 1965, 2, 285–290.
- Koszul, J.L. Déformations des variétés localement plates. Ann. Inst. Fourier 1968, 18, 103–114.
- Koszul, J.L. Trajectoires Convexes de Groupes Affines Unimodulaires. In Essays on Topology and Related Topics; Springer: Berlin, Germany, 1970; pp. 105–110.
- Vey, J. Sur les Automorphismes Affines des Ouverts Convexes Saillants. Annali della Scuola Normale Superiore di Pisa, Classe di Science 1970, 24, 641–665.

Hirohiko Shima, Emeritus Professor of Yamaguchi university, Phd Student at Osaka University

- H. Shima, "The Geometry of Hessian Structures", World Scientific, 2007
- http://www.worldscientific.com/worldscibooks/10.1142/6241
  dedicated to Prof. Jean-Louis Koszul (« the content of the present book finds their origin in his studies »)
- H. Shima,Symmetric spaces with invariant locally Hessian structures, J. Math. Soc. Japan,, pp. 581-589., 1977
- H. Shima, « Homogeneous Hessian manifolds », Ann. Inst. Fourier, Grenoble, pp. 91-128., 1980
- H. Shima, « Vanishing theorems for compact Hessian manifolds », Ann. Inst. Fourier, Grenoble, pp.183-205., 1986
- H. Shima, « Harmonicity of gradient mappings of level surfaces in a real affine space », Geometriae Dedicata, pp. 177-184., 1995
- H. Shima, « Hessian manifolds of constant Hessian sectional curvature », J. Math. Soc. Japan, pp. 735-753., 1995
- H. Shima, « Homogeneous spaces with invariant projectively flat affine connections », Trans. Amer. Math. Soc., pp. 4713-4726, 1999

Prof. Michel Nguiffo Boyom tutorial : Géométrie de l'Information
http://repmus.ircam.fr/_media/brillouin/ressources/une-source-de-nouveaux-invariants-de-la-geometrie-de-l-information.pdf

Jean-Louis Koszul and Hirohiko Shima



Jean-Louis Koszul and Hirohiko Shima
2013, Ecole des Mines de Paris
GSI'13 « Geometric Science of Information »

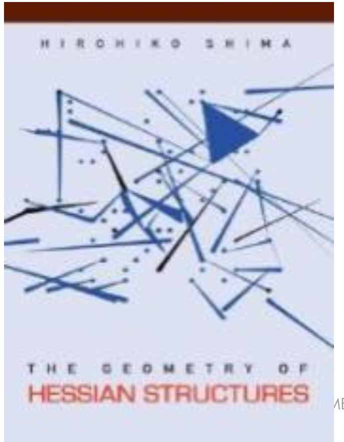
THALES

Jean-Louis Koszul and hessian structures

Hessian Geometry of Jean-Louis Koszul

- Hirohiko Shima Book, « Geometry of Hessian Structures », world Scientific Publishing 2007, dedicated to Jean-Louis Koszul
- Hirohiko Shima Keynote Talk at GSI'13
  - http://www.see.asso.fr/file/5104/download/9914
- Prof. M. Boyom tutorial :
  - http://repmus.ircam.fr/_media/brillouin/ressources/une-source-de-nouveaux-invariants-de-la-geometrie-de-l-information.pdf

Jean-Louis Koszul



J.L. Koszul, « Sur la forme hermitienne canonique des espaces homogènes complexes », Canad. J. Math. 7, pp. 562-576., 1955
J.L. Koszul, « Domaines bornées homogènes et orbites de groupes de transformations affines », Bull. Soc. Math. France 89, pp. 515-533., 1961
J.L. Koszul, « Ouverts convexes homogènes des espaces affines », Math. Z. 79, pp. 254-259., 1962
J.L. Koszul, « Variétés localement plates et convexité », Osaka J. Maht. 2, pp. 285-290., 1965
J.L. Koszul, « Déformations des variétés localement plates », .Ann Inst Fourier, 18 , 103-114., 1968

Study of Koszul Works by Michel Nguiffo Boyom

Michel Nguiffo Boyom (Institut Montpelierain Alexander Grothendieck)

- Bridges between differential topology (Feuilletages), and Fisher metric of Information Geometry have their pillars in **KV (Koszul-Vinberg) cohomology of locally flat manifolds and in Koszul convexity**
- Connexions are objects of Information Geometry. They are **deformations in space of linear connexions of Koszul connexion that control the geometry of statistical model**. These connexions are parameterized by **2-cochaines of KV(Koszul-Vinberg) complexe**. Fisher connexion is one of them. Curvature of Linear Connexion is a value of Maurer-Cartan Polynomial. One raison in favor of this polynomial is that it **controles all locally flat structures** via anomalies theory. … Nijenhuis, Koszul, Gerstenhaber, Vinberg and soviet school have initiated **the study of the connexion locally flat**.
- Notion of **convexity of locally flat manifolds** is introduced by Jean-Louis Koszul. Considerations that drive this approach find their roots in **Geometry of bounded homogeneous manafolds**. Convexity notion introduced by Koszul is the real analogue of holomorph convexity of Kaup
- Prof. **Michel Nguiffo Boyom** tutorial « Géométrie de l'Information » : http://repmus.ircam.fr/_media/brillouin/ressources/une-source-de-nouveaux-invariants-de-la-geometrie-de-l-information.pdf

Koszul-Vinberg Characteristic Function/Metric of convex cone

- J.L. Koszul and E. Vinberg have introduced an affinely invariant Hessian metric on a sharp convex cone through its characteristic function.

- Ω is a sharp open convex cone in a vector space E of finite dimension on R (a convex cone is sharp if it does not contain any full straight line).

- Ω^* is the dual cone of Ω and is a sharp open convex cone.

- Let $d\xi$ the Lebesgue measure on E^* dual space of E, the following integral:

$$\psi_\Omega(x) = \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi \quad \forall x \in \Omega$$

is called the **Koszul-Vinberg characteristic function**

Koszul-Vinberg Characteristic Function/Metric of convex cone

- Koszul-Vinberg Metric : $g = d^2 \log \psi_\Omega$

$$d^2 \log \psi(x) = d^2 \left[\log \int \psi_u du\right] = \frac{\int \psi_u d^2 \log \psi_u du}{\int \psi_u du} + \frac{1}{2} \frac{\iint \psi_u \psi_v (d \log \psi_u - d \log \psi_v)^2 dudv}{\iint \psi_u \psi_v dudv}$$

- We can define a diffeomorphism by: $x^* = -\alpha_x = -d \log \psi_\Omega(x)$

with $\langle df(x), u \rangle = D_u f(x) = \left. \frac{d}{dt} \right|_{t=0} f(x+tu)$

- When the cone Ω is symmetric, the map $x^* = -\alpha_x$ is a bijection and an isometry with a unique fixed point (the manifold is a Riemannian Symmetric Space given by this isometry):

$$(x^*)^* = x \quad \langle x, x^* \rangle = n \qquad \psi_\Omega(x) \psi_{\Omega^*}(x^*) = cste$$

- x^* is characterized by $x^* = \arg\min \{ \psi(y) / y \in \Omega^*, \langle x, y \rangle = n \}$

- x^* is the center of gravity of the cross section $\{ y \in \Omega^*, \langle x, y \rangle = n \}$ of : Ω^*

$$x^* = \int_{\Omega^*} \xi . e^{-\langle \xi, x \rangle} d\xi \Big/ \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi$$

THALES

Koszul Entropy via Legendre Transform

- we can deduce "Koszul Entropy" defined as Legendre Transform of (-)Koszul-Vinberg characteristic function $\Phi(x) = -\log \psi_\Omega(x)$:

$$\Phi^*(x^*) = \left\langle x, x^* \right\rangle - \Phi(x)$$

with $x^* = D_x \Phi$ and $x = D_{x^*} \Phi^*$ where $\psi_\Omega(x) = \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi \quad \forall x \in \Omega$

- Demonstration: we set $x^* = \int_{\Omega^*} \xi . e^{-\langle \xi, x \rangle} d\xi / \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi$

Using $\left\langle -x^*, h \right\rangle = d_h \log \psi_\Omega(x) = -\int_{\Omega^*} \langle \xi, h \rangle e^{-\langle \xi, x \rangle} d\xi / \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi$

and $-\left\langle x^*, x \right\rangle = \int_{\Omega^*} \log e^{-\langle \xi, x \rangle} . e^{-\langle \xi, x \rangle} d\xi / \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi$

$$\Phi^*(x^*) = -\int_{\Omega^*} \log e^{-\langle \xi, x \rangle} . e^{-\langle \xi, x \rangle} d\xi / \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi + \log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi$$

$$\Phi^*(x^*) = \left[\left(\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi \right) . \log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi - \int_{\Omega^*} \log e^{-\langle \xi, x \rangle} . e^{-\langle \xi, x \rangle} d\xi \right] / \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi$$

Koszul-Vinberg Characteristic Function Legendre Transform

$$\Phi^*(x^*) = \langle x, x^* \rangle - \Phi(x) = -\int_{\Omega^*} \log e^{-\langle \xi, x \rangle} . e^{-\langle \xi, x \rangle} d\xi \Big/ \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi + \log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi$$

$$\Phi^*(x^*) = \left[\left(\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi \right) . \log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi - \int_{\Omega^*} \log e^{-\langle \xi, x \rangle} . e^{-\langle \xi, x \rangle} d\xi \right] \Big/ \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi$$

$$\Phi^*(x^*) = \left[\log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi - \int_{\Omega^*} \log e^{-\langle \xi, x \rangle} . \frac{e^{-\langle \xi, x \rangle}}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi} d\xi \right]$$

$$\Phi^*(x^*) = \left[\log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi . \left(\int_{\Omega^*} \frac{e^{-\langle \xi, x \rangle}}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi} d\xi \right) - \int_{\Omega^*} \log e^{-\langle \xi, x \rangle} . \frac{e^{-\langle \xi, x \rangle}}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi} d\xi \right] \text{ with } \int_{\Omega^*} \frac{e^{-\langle \xi, x \rangle}}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi} d\xi = 1$$

$$\Phi^*(x^*) = \left[-\int_{\Omega^*} \frac{e^{-\langle \xi, x \rangle}}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi} . \log \left(\frac{e^{-\langle \xi, x \rangle}}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi} \right) d\xi \right]$$

SHANNON ENTROPY

Koszul Entropy via Legendre Transform

We can consider this Legendre transform as an entropy:

$$\Phi^* = -\int_{\Omega^*} \frac{e^{-\langle \xi, x\rangle}}{\int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi} \log \frac{e^{-\langle \xi, x\rangle}}{\int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi} d\xi = -\int_{\Omega^*} p_x(\xi) \log p_x(\xi) d\xi$$

With $$p_x(\xi) = e^{-\langle \xi, x\rangle} / \int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi = e^{-\langle x,\xi\rangle - \log \int_{\Omega^*} e^{-\langle \xi,x\rangle} d\xi} = e^{-\langle x,\xi\rangle + \Phi(x)}$$

and $$x^* = D_x\Phi = \int_{\Omega^*} \xi . p_x(\xi) d\xi = \int_{\Omega^*} \xi . e^{-\langle x,\xi\rangle + \Phi(x)} d\xi = \int_{\Omega^*} \xi . e^{-\Phi^*(\xi)} d\xi$$

$$\Phi(x) = -\log \int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi = -\log \int_{\Omega^*} e^{-[\Phi^*(\xi) + \Phi(x)]} d\xi$$

$$\Phi(x) = \Phi(x) - \log \int_{\Omega^*} e^{-\Phi^*(\xi)} d\xi \Rightarrow \int_{\Omega^*} e^{-\Phi^*(\xi)} d\xi = 1$$

$$\text{Jensen Ineq.}: \Phi^* \text{ conv.} \Rightarrow \Phi^*(E[\xi]) \leq E[\Phi^*(\xi)]$$

$$\text{Legendre Transform}: \Phi^*(x^*) \geq \langle x, x^*\rangle - \Phi(x)$$

$$\Rightarrow \Phi^*(x^*) \geq \int_{\Omega^*} \Phi^*(\xi) p_x(\xi) d\xi = E[\Phi^*(\xi)]$$

$$\log p_x(\xi) = \log e^{-\langle x,\xi\rangle + \Phi(x)} = \log e^{-\Phi^*(\xi)} = -\Phi^*(\xi)$$

$$\Rightarrow -\int_{\Omega^*} \log p_x(\xi) p_x(\xi) d\xi = \int_{\Omega^*} \Phi^*(\xi) p_x(\xi) d\xi = \Phi^*(x^*)$$

$$\text{if and only if } \int_{\Omega^*} \Phi^*(\xi) p_x(\xi) d\xi = \Phi^*\left(\int_{\Omega^*} \xi . p_x(\xi) d\xi\right)$$

$$\text{or } E[\Phi^*(\xi)] = \Phi^*(E[\xi])$$

OPEN

Barycenter of Koszul Entropy = Koszul Entropy of Barycenter

$$E\left[\Phi^*(\xi)\right]=\Phi^*(E[\xi])$$

$$\int_{\Omega^*}\Phi^*(\xi)p_x(\xi)d\xi=\Phi^*\left(\int_{\Omega^*}\xi.p_x(\xi)d\xi\right)$$

$$p_x(\xi)=e^{-\langle\xi,x\rangle}/\int_{\Omega^*}e^{-\langle\xi,x\rangle}d\xi=e^{-\langle x,\xi\rangle-\log\int_{\Omega^*}e^{-\langle\xi,x\rangle}d\xi}=e^{-\langle x,\xi\rangle+\Phi(x)}$$

$$x^*=D_x\Phi=\int_{\Omega^*}\xi.p_x(\xi)d\xi=\int_{\Omega^*}\xi.e^{-\langle x,\xi\rangle+\Phi(x)}d\xi=\int_{\Omega^*}\xi.e^{-\Phi^*(\xi)}d\xi$$

$$\Phi^*(x^*)=\underset{x}{Sup}\left[\langle x,x^*\rangle-\Phi(x)\right]$$

Koszul metric & Fisher Metric

- To make the link with Fisher metric given by matrix $I(x)$, we can observe that the second derivative of $\log p_x(\xi)$ is given by:

$$\log p_x(\xi) = -\Phi^*(\xi) = \Phi(x) - \langle x, \xi \rangle$$

$$\frac{\partial^2 \log p_x(\xi)}{\partial x^2} = \frac{\partial^2 \left[\Phi(x) - \langle x, \xi \rangle\right]}{\partial x^2} = \frac{\partial^2 \Phi(x)}{\partial x^2}$$

$$\Rightarrow I(x) = -E_\xi\left[\frac{\partial^2 \log p_x(\xi)}{\partial x^2}\right] = -\frac{\partial^2 \Phi(x)}{\partial x^2} = \frac{\partial^2 \log \psi_\Omega(x)}{\partial x^2}$$

- We could then deduce the close interrelation between Fisher metric and hessian of Koszul-Vinberg characteristic logarithm.

$$I(x) = -E_\xi\left[\frac{\partial^2 \log p_x(\xi)}{\partial x^2}\right] = \frac{\partial^2 \log \psi_\Omega(x)}{\partial x^2}$$

Koszul Metric and Fisher Metric as Variance

- We can also observed that the Fisher metric or hessian of KVCF logarithm is related to the variance of ξ :

$$\log \Psi_\Omega(x) = \log \int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi \Rightarrow \frac{\partial \log \Psi_\Omega(x)}{\partial x} = -\frac{1}{\int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi} \int_{\Omega^*} \xi . e^{-\langle \xi, x\rangle} d\xi$$

$$\frac{\partial^2 \log \Psi_\Omega(x)}{\partial x^2} = -\frac{1}{\left(\int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi\right)^2} \left[-\int_{\Omega^*} \xi^2 . e^{-\langle \xi, x\rangle} d\xi . \int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi + \left(\int_{\Omega^*} \xi . e^{-\langle \xi, x\rangle} d\xi\right)^2 \right]$$

$$\frac{\partial^2 \log \Psi_\Omega(x)}{\partial x^2} = \int_{\Omega^*} \xi^2 . \frac{e^{-\langle \xi, x\rangle}}{\int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi} d\xi - \left(\int_{\Omega^*} \xi . \frac{e^{-\langle \xi, x\rangle}}{\int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi} d\xi \right)^2 = \int_{\Omega^*} \xi^2 . p_x(\xi) d\xi - \left(\int_{\Omega^*} \xi . p_x(\xi) d\xi \right)^2$$

$$I(x) = -E_\xi\left[\frac{\partial^2 \log p_x(\xi)}{\partial x^2}\right] = \frac{\partial^2 \log \psi_\Omega(x)}{\partial x^2} = E_\xi\left[\xi^2\right] - E_\xi\left[\xi\right]^2 = Var(\xi)$$

Definition of Maximum Entropy Density

- How to replace x by mean value of ξ , $\bar{\xi}(=x^*)$ in :

$$p_x(\xi)=\frac{e^{-\langle \xi,x\rangle}}{\int\limits_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi} \quad \text{with} \quad \bar{\xi}=\int\limits_{\Omega^*}\xi.p_x(\xi)d\xi$$

- Legendre Transform will do this inversion by inversing $\bar{\xi}=\frac{d\Phi(x)}{dx}$

- We then observe that Koszul Entropy provides density of Maximum Entropy with this general definition of density:

$$p_{\bar{\xi}}(\xi)=\frac{e^{-\left\langle \xi,\Theta^{-1}\left(\bar{\xi}\right)\right\rangle}}{\int\limits_{\Omega^*} e^{-\left\langle \xi,\Theta^{-1}\left(\bar{\xi}\right)\right\rangle}d\xi}$$

$$x=\Theta^{-1}(\bar{\xi})$$

$$\bar{\xi}=\Theta(x)=\frac{d\Phi(x)}{dx}$$

where $\bar{\xi}=\int\limits_{\Omega^*}\xi.p_{\bar{\xi}}(\xi)d\xi$ and $\Phi(x)=-\log\int\limits_{\Omega^*} e^{-\langle x,\xi\rangle}d\xi$

Cartan-Killing Form and Invariant Inner Product

- It is not possible to define an ad(g)-invariant inner product for any two elements of a Lie Algebra, but a symmetric bilinear form, called "Cartan-Killing form", could be introduced (Elie Cartan PhD 1894)

- This form is defined according to the adjoint endomorphism ad_x of $\mathfrak{g}$ that is defined for every element x of $\mathfrak{g}$ with the help of the Lie bracket: $ad_x(y) = [x, y]$

- The trace of the composition of two such endomorphisms defines a bilinear form, the Cartan-Killing form: $B(x, y) = Tr(ad_x ad_y)$

- The Cartan-Killing form is symmetric: $B(x, y) = B(y, x)$

- and has the associativity property: $B([x, y], z) = B(x, [y, z])$

- given by:

$$B([x, y], z) = Tr(ad_{[x,y]} ad_z) = Tr([ad_x, ad_y] ad_z)$$
$$B([x, y], z) = Tr(ad_x [ad_y, ad_z]) = B(x, [y, z])$$

THALES

Cartan-Killing Form and Invariant Inner Product

- Elie Cartan has proved that if $\mathfrak{g}$ is a simple Lie algebra (the Killing form is non-degenerate) then **any invariant symmetric bilinear form on $\mathfrak{g}$ is a scalar multiple of the Cartan-Killing form.**

- The Cartan-Killing form is invariant under automorphisms $\sigma \in Aut(\mathfrak{g})$ of the algebra $\mathfrak{g}$: $B(\sigma(x),\sigma(y)) = B(x,y)$

- To prove this invariance, we have to consider:

$$\begin{cases} \sigma[x,y] = [\sigma(x),\sigma(y)] \\ z = \sigma(y) \end{cases} \Rightarrow \sigma[x,\sigma^{-1}(z)] = [\sigma(x),z]$$

rewritten $ad_{\sigma(x)} = \sigma \circ ad_x \circ \sigma^{-1}$

$$B(\sigma(x),\sigma(y)) = Tr(ad_{\sigma(x)} ad_{\sigma(y)}) = Tr(\sigma \circ ad_x ad_y \circ \sigma^{-1})$$
$$B(\sigma(x),\sigma(y)) = Tr(ad_x ad_y) = B(x,y)$$

Cartan-Killing Form and Invariant Inner Product

A natural G-invariant inner product could be introduced by Cartan-Killing form:

- Cartan Generating Inner Product: The following Inner product defined by Cartan-Killing form is invariant by automorphisms of the algebra

$$\langle x, y \rangle = -B(x, \theta(y))$$

where $\theta \in \mathfrak{g}$ is a Cartan involution (An involution on $\mathfrak{g}$ is a Lie algebra automorphism θ of $\mathfrak{g}$ whose square is equal to the identity).

From Cartan-Killing Form to Koszul Information Metric

$$B(x,y)=Tr\left(ad_x ad_y\right)$$

Cartan – Killing Form

$$\langle x,y\rangle=-B(x,\theta(y))$$

with $\theta\in g$, Cartan Involution

Koszul Characteristic Function

$$\Phi(x)=-\log\int_{\Omega^*}e^{-\langle\xi,x\rangle}d\xi\quad\forall x\in\Omega$$

Koszul Entropy

$$\Phi^*(x^*)=\langle x,x^*\rangle-\Phi(x)$$

$$\Phi^*(x^*)=-\int_{\Omega^*}p_x(\xi)\log p_x(\xi)d\xi$$

with $x^*=\int_{\Omega^*}\xi.p_x(\xi)d\xi$

Koszul Density

$$p_x(\xi)=\frac{e^{-\langle\xi,x\rangle}}{\int_{\Omega^*}e^{-\langle\xi,x\rangle}d\xi}$$

Koszul Metric

$$I(x)=-E_\xi\left[\frac{\partial^2\log p_x(\xi)}{\partial x^2}\right]$$

$$I(x)==-\frac{\partial^2\Phi(x)}{\partial x^2}=\frac{\partial^2\log\int_{\Omega^*}e^{-\langle\xi,x\rangle}d\xi}{\partial x^2}$$

Relation of Koszul density with Maximum Entropy Principle

- The density from Maximum Entropy Principle is given by:

$$\underset{p_x(.)}{Max}\left[-\int_{\Omega^*} p_x(\xi)\log p_x(\xi)d\xi\right] \text{ such } \begin{cases}\int_{\Omega^*} p_x(\xi)d\xi = 1 \\ \int_{\Omega^*} \xi.p_x(\xi)d\xi = x^*\end{cases}$$

- If we take $q_x(\xi) = e^{-\langle \xi,x\rangle} / \int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi = e^{-\langle x,\xi\rangle - \log\int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi}$ such that

$$\begin{cases}\int_{\Omega^*} q_x(\xi).d\xi = \int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi / \int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi = 1 \\ \log q_x(\xi) = \log e^{-\langle x,\xi\rangle - \log\int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi} = -\langle x,\xi\rangle - \log\int_{\Omega^*} e^{-\langle x,\xi\rangle}d\xi\end{cases}$$

- Then by using the fact that $\log x \geq \left(1 - x^{-1}\right)$ with equality if and only if , $x = 1$, we find the following:

$$-\int_{\Omega^*} p_x(\xi)\log\frac{p_x(\xi)}{q_x(\xi)}d\xi \leq -\int_{\Omega^*} p_x(\xi)\left(1 - \frac{q_x(\xi)}{p_x(\xi)}\right)d\xi$$

THALES

Relation of Koszul density with Maximum Entropy Principle

- **We can then observe that:**

$$\int_{\Omega^*} p_x(\xi)\left(1-\frac{q_x(\xi)}{p_x(\xi)}\right)d\xi = \int_{\Omega^*} p_x(\xi)d\xi - \int_{\Omega^*} q_x(\xi)d\xi = 0$$

because $\int_{\Omega^*} p_x(\xi)d\xi = \int_{\Omega^*} q_x(\xi)d\xi = 1$

- **We can then deduce that:**

$$\int_{\Omega^*} p_x(\xi)\log\frac{p_x(\xi)}{q_x(\xi)}d\xi \leq 0 \Rightarrow -\int_{\Omega^*} p_x(\xi)\log p_x(\xi)d\xi \leq -\int_{\Omega^*} p_x(\xi)\log q_x(\xi)d\xi$$

- **If we develop the last inequality, using expression of $q_x(\xi)$:**

$$-\int_{\Omega^*} p_x(\xi)\log p_x(\xi)d\xi \leq -\int_{\Omega^*} p_x(\xi)\left[-\langle x,\xi\rangle - \log\int_{\Omega^*} e^{-\langle x,\xi\rangle}d\xi\right]d\xi$$

$$-\int_{\Omega^*} p_x(\xi)\log p_x(\xi)d\xi \leq \left\langle x, \int_{\Omega^*} \xi.p_x(\xi)d\xi\right\rangle + \log\int_{\Omega^*} e^{-\langle x,\xi\rangle}d\xi$$

$$-\int_{\Omega^*} p_x(\xi)\log p_x(\xi)d\xi \leq \langle x, x^*\rangle - \Phi(x)$$

$$-\int_{\Omega^*} p_x(\xi)\log p_x(\xi)d\xi \leq \Phi^*(x^*)$$

General Theory: Koszul-Vey Theorem

J.L. Koszul and J. Vey have proved the following theorem:

- Koszul J.L., Variétés localement plates et convexité, Osaka J. Math.,n°2 ,p.285-290, 1965
- Vey J., Sur les automorphismes affines des ouverts convexes saillants, Annali della Scuola Normale Superiore di Pisa, Classe di Science, 3e série, tome 24,n°4, p.641-665, 1970

Koszul-Vey Theorem:

Let M be a connected Hessian manifold with Hessian metric g .
Suppose that admits a closed 1-form α such that $D\alpha = g$ and
there exists a group G of affine automorphisms of M preserving α:

- If M/G is quasi-compact, then the universal covering manifold of M is affinely isomorphic to a convex domain Ω real affine space not containing any full straight line.
- If M/G is compact, then Ω is a sharp convex cone.

- Koszul J.L., Variétés localement plates et convexité, Osaka J. Math. , n°2, p.285-290, 1965
- Vey J., Sur les automorphismes affines des ouverts convexes saillants, Annali della Scuola Normale Superiore di Pisa, Classe di Science, 3e série, tome 24,n°4, p.641-665, 1970

Koszul Forms/Metric for Homogenous Siegel Domains SD

Koszul Forms for Homogeneous Bounded domains

- Koszul has developed his previously described theory for Homogenous Siegel Domains *SD*. He has proved that there is a subgroup *G* in the group of the complex affine automorphisms of these domains (Iwasawa subgroup), such that *G* acts on SD simply transitively. The Lie algebra $\mathfrak{g}$ of *G* has a structure that is an algebraic translation of the Kähler structure of *SD.*
- There is an integrable almost complex structure *J* on $\mathfrak{g}$ and there exists $\eta \in \mathfrak{g}^*$ such that $\langle X,Y \rangle_\eta = \langle [JX,Y], \eta \rangle$ defines a *J*-invariant positive definite inner product on $\mathfrak{g}$. Koszul has proposed as admissible form $\eta \in \mathfrak{g}^*$, the form ξ :

$$\Psi(X) = \langle X, \xi \rangle = Tr[ad(JX) - J.ad(X)] \quad \forall X \in \mathfrak{g}$$

- Koszul has proved that $\langle X,Y \rangle_\xi$ coincides, up to a positive number multiple with the real part of the Hermitian inner product obtained by the Bergman metric of SD by identifying $\mathfrak{g}$ with the tangent space of SD. The Koszul forms are then given by:

$$\alpha = -\frac{1}{4} d\Psi(X) \qquad \beta = D\alpha$$

Koszul Forms/Metric for Homogenous Siegel Domains SD

Koszul Forms

- 1st Koszul Form : $\alpha = -\frac{1}{4} d\Psi(X)$

$$\Psi(X) = Tr_{g/b}\left[ad(JX) - Jad(X)\right] \quad \forall X \in \mathfrak{g}$$

- 2nd Koszul Form: $\beta = D\alpha$

Application for Poincaré Upper-Hal Plane:

$$V = \{z = x + iy \,/\, y > 0\} \qquad Y = y\frac{d}{dy} \quad \Rightarrow \begin{cases} ad(Y).Z = [Y,Z] \\ [X,Y] = -Y \\ JX = Y \end{cases}$$

With $X = y\frac{d}{dx}$ and $\begin{cases} Tr[ad(JX) - Jad(X)] = 2 \\ Tr[ad(JY) - Jad(Y)] = 0 \end{cases}$

We can deduce that

$$\Psi(X) = 2\frac{dx}{y} \Rightarrow \alpha = -\frac{1}{4} d\Psi = -\frac{1}{2}\frac{dx \wedge dy}{y^2}$$

$$\Rightarrow ds^2 = \frac{dx^2 + dy^2}{2y^2} \qquad \Omega = \frac{1}{y^2} dx \wedge dy$$


J. L. KOSZUL

EXPOSÉS SUR LES ESPACES HOMOGENES SYMÉTRIQUES

PUBLICAÇÃO DA SOCIEDADE DE MATEMÁTICA DE SÃO PAULO

SÃO PAULO

1959

Jean-Louis Koszul Forms for Siegel Upper-Half Plane

Koszul form for Siegel Upper-Half Plane: $V=\{Z=X+iY\,/\,Y>0\}$

- Symplectic Group :

$$\begin{cases} SZ=(AZ+B)D^{-1} \\ A^TD=I,\, B^TD=D^TB \end{cases} \quad \text{with} \quad S=\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \quad \text{and} \quad J=\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \text{ with } \begin{cases} b=b^T \\ d=-a^T \end{cases} \text{ and base } \alpha_{ij}=\begin{pmatrix} e_{ij} & 0 \\ 0 & -e_{ji} \end{pmatrix},\ \beta_{ij}=\begin{pmatrix} 0 & e_{ij}+e_{ji} \\ 0 & 0 \end{pmatrix}$$

$$\begin{cases} \Psi(\alpha_{ij})=0 \\ \Psi(\beta_{ij})=\delta_{ij}(3p+1) \end{cases} \Rightarrow \begin{cases} \Psi(dX+idY)=\dfrac{3p+1}{2}Tr\left(Y^{-1}dX\right) \\ \Omega=-\dfrac{1}{4}d\Psi=\dfrac{3P+1}{8}Tr\left(Y^{-1}dZ\wedge Y^{-1}d\bar{Z}\right) \\ ds^2=\dfrac{(3p+1)}{8}Tr\left(Y^{-1}dZY^{-1}d\bar{Z}\right) \end{cases}$$

Filiation Poincaré/Cartan/Koszul

« Il est clair que si l'on parvenait à démontrer que tous les domaines homogènes dont la forme

$$\Phi = \sum_{i,j} \frac{\partial^2 \log K(z,\bar{z})}{\partial z_i \partial \bar{z}_j} dz_i d\bar{z}_j$$

est définie positive sont symétriques, toute la théorie des domaines bornés homogènes serait élucidée. C'est là un problème de géométrie hermitienne certainement très intéressant »

Dernière phrase de Elie Cartan, dans « Sur les domaines bornés de l'espace de n variables complexes », Abh. Math. Seminar Hamburg, 1935

Carl Ludwig Siegel

(Siegel space of 1st king and Symplectic Geometry)

Lookeng Hua

(Bergman Kernel, Cauchy and Poisson of Siegel Domains

Ernest Vinberg

(Siegel Domains of 2nd kind)

Structure of Information Geometry (Koszul Hessian Geometry)

Henri Poincaré

(half-plane) n=1

Elie Cartan

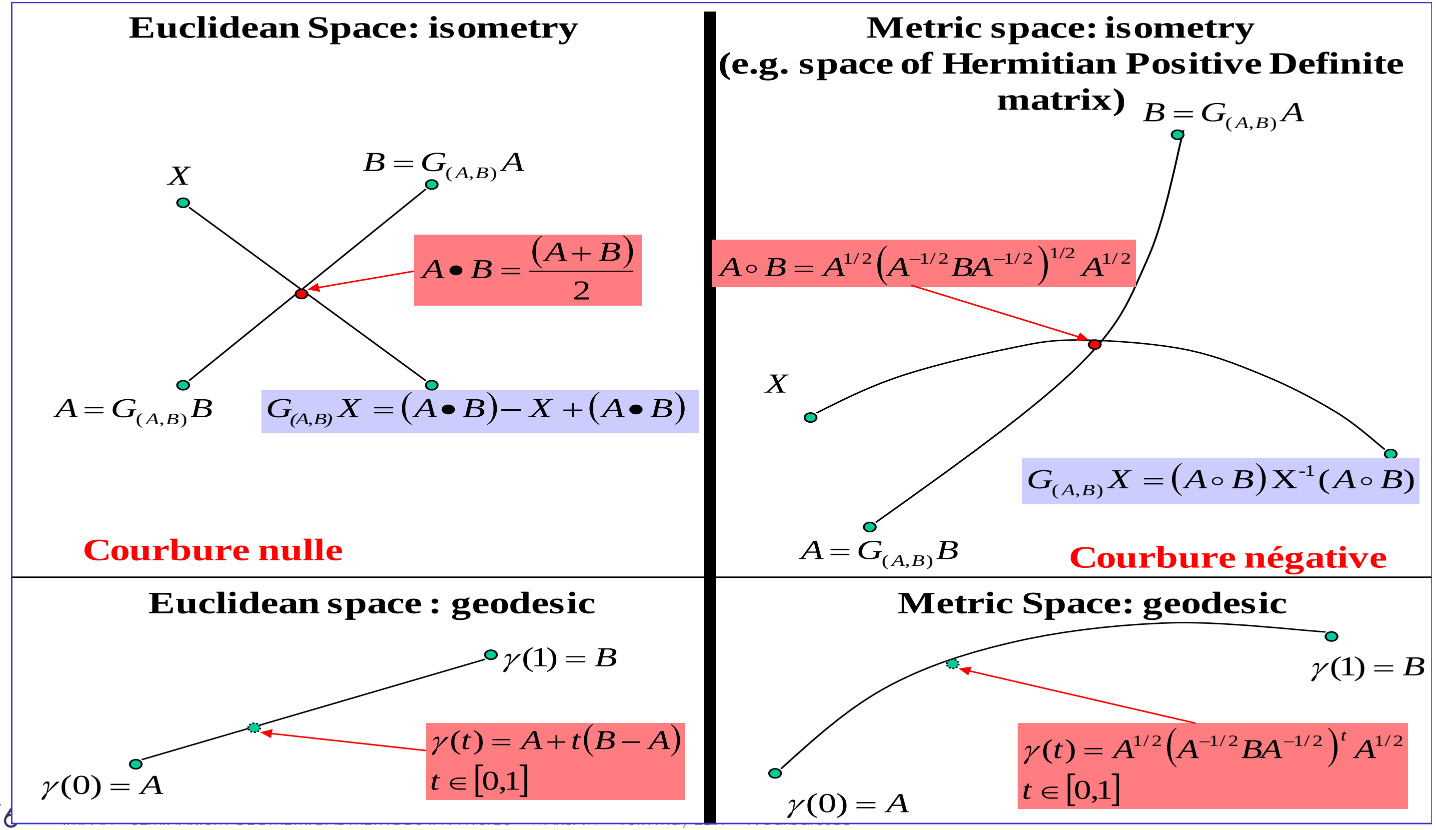
(classification in 6 classes of symmetric homogeneous bounded domains) n<=3

Jean-Louis Koszul

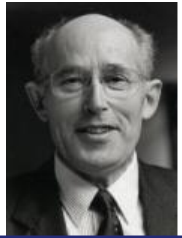
(hermitian canonical forms of complex homogeneous spaces, a complex homogenesous space with positive definite hermitian canonical form is isomorphe to a bounded domain, study of affine transformation groups of locally flat manifolds)

Study of Symmetric Spaces and their classification

$$G_{(A,B)}X = (A \circ B)X^{-1}(A \circ B) \quad \text{avec} \quad A \circ B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2}A^{1/2}$$




E. J. Cartan


M. Berger

THALES

METRIC SPACE EVERYWHERE: Metric given by Fisher Matrix and Information Geometry

Seminal Papers on Information Geometry

Information Geometry:

- Cramer-Rao-Fréchet-Darmois Bound and Fisher Information Matrix

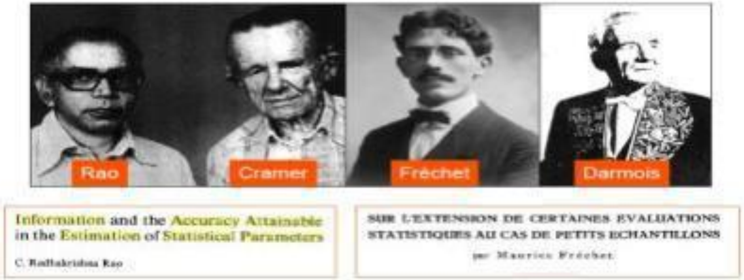
CRFD Bound

$$E\left[\left(\theta-\hat{\theta}\right)\left(\theta-\hat{\theta}\right)^{+}\right]\geq I(\theta)^{-1}$$

Fisher Information Matrix

$$\left[I(\theta)\right]_{i,j}=-E\left[\frac{\partial^2 \ln p(X/\theta)}{\partial\theta_i\partial\theta_j^*}\right]$$

R.A. Fisher



1945

1943
(IHP Lecture 1939)

- Kulback-Leibler Divergence (variational definition by Donsker/Varadhan) :

$$K(p,q)=\underset{\phi}{Sup}\left[E_p(\phi)-\ln E_q\left(e^{\phi}\right)\right]=\int p(x/\theta)\ln\left(\frac{p(x/\theta)}{q(x/\theta)}\right)dx$$

- Rao-Chentsov Metric (invariance by non-singular parameterization change)

$$ds^2=K\left[p(X/\theta),p(X/\theta+d\theta)\right]=d\theta^{+}I(\theta)d\theta=\sum_{i,j}g_{i,j}d\theta_i d\theta_j^*$$

N. N. Chentsov

- Invariance: $w=W(\theta)\Rightarrow ds^2(w)=ds^2(\theta)$

THA...

Maurice Fréchet : IHP Lecture 1939, Paper 1943

- The Inverse of the Fisher/Information Matrix defines the lower bound of statistical estimators. Classically, this Lower bound is called Cramer-Rao Bound because it was described in the Rao's paper of 1945. Historically, this bound has been <u>published first by Maurice Fréchet in 1939 in his winter "Mathematical Statistics" Lecture at the Institut Henri Poincaré during winter 1939–1940</u>. Maurice Fréchet has published these elements in a paper as early as 1943. We can read at the bottom of the first page of his paper:

Fréchet, M. Sur l'extension de certaines évaluations statistiques au cas de petits échantillons. Revue de l'Institut International de Statistique 1943, 11, 182–205.

- At the bottom of 1st page of Fréchet's paper, we can read:
  - The contents of this report formed a part of <u>our lecture of mathematical statistics at the Henri Poincaré institute during winter 1939–1940</u>. It constitutes one of the chapters of the second exercise book (in preparation) of our "Lessons of Mathematical Statistics", the first exercise book of which, "Introduction: preliminary Presentation of Probability theory" (119 pages quarto, typed) has just been published in the "Centre de Documentation Universitaire, Tournois et Constans. Paris".

1943 Fréchet Paper: « Distinguished functions » & « Clairaut Equation »

- In 1943, Maurice Fréchet wrote a seminal paper introducing what was then called the Cramer-Rao bound.

$$(\sigma_T)^2 \geqslant \frac{1}{n(\sigma_A)^2} \text{ avec } T = H(X_1, \ldots, X_n), \quad A = \frac{1}{f(X,\theta)} \frac{\partial f(X,\theta)}{\partial \theta}$$

- This paper contains in fact much more that this important discovery. In particular, Maurice Fréchet introduces more general notions relative to "distinguished functions", densities with estimator reaching the bound, defined with a function, solution of Clairaut's equation. The solutions "envelope of the Clairaut's equation" are equivalents to standard Legendre transform with only smoothness assumption.

$$(55) \qquad \mu = \theta\,\mu' - \psi(\mu')$$

c'est-à-dire une équation de Clairaut. La solution μ' = constante réduirait $f(x, \theta)$, d'après (48) à une fonction indépendante de θ, cas où le problème n'aurait plus de sens. μ est donc donné par la solution singulière de (55), qui est unique et s'obtient en éliminant s entre $\mu = \theta\, s - \psi(s)$ et $\theta = \psi'(s)$ ou encore entre

1943 Fréchet Paper: « Distinguished functions » & « Clairaut Equation »

Etude des densités distinguées. Appelons (provisoirement, dans ce mémoire) *densité distinguée*, toute densité de probabilité $f(x, \theta)$ telle que la fonction

(46) $$\theta + \frac{\dfrac{\partial L f(x,\theta)}{\partial \theta}}{\displaystyle\int_{-\infty}^{+\infty} \left[\frac{\partial}{\partial \theta} f(x,\theta)\right]^2 \frac{dx}{f(x,\theta)}}$$

soit indépendante de θ.

Pour ces densités distinguées, on va pouvoir déterminer la fonction minimisante $H'(X_1, \ldots X_n)$ et étendre au cas des petits échantillons la comparaison des méthodes d'estimation faites par divers auteurs dans le cas des grands échantillons. Il vaut donc la peine de chercher la forme générale de $f(x, \theta)$ pour cette catégorie de variables.

En appelant $h(x)$ cette fonction, on voit qu'on a l'identité de la forme

(47) $$\lambda(\theta) \frac{\partial}{\partial \theta} L f(x,\theta) = h(x) - \theta$$

où $\lambda(\theta) > 0$. On peut considérer $\frac{1}{\lambda(\theta)}$ comme la dérivée seconde d'une fonction $\mu(\theta)$; d'où $\frac{\partial}{\partial \theta} L f(x,\theta) = \mu_\Theta''(\theta)\ [h(x) - \theta]$.

Par suite $L f(x, \theta) - \mu'_\Theta\ [h(x) - \theta] - \mu(\theta)$ est une quantité indépendante de θ que nous pouvons représenter par $l(x)$.

Ainsi toute densité distinguée, $f(x, \theta)$, est de la forme

(48) $$f(x,\theta) = e^{\mu'_\Theta [h(x) - \Theta] + \mu(\Theta) + l(x)}$$

(52bis) $$\lambda\, \mu'' = 1.$$

Incidemment, puisque, d'après (52), $\lambda(\theta)$ est positif, il en résulte aussi que $\mu''\left(= \frac{1}{\lambda(\theta)}\right)$ est aussi positif.

On peut d'ailleurs préciser d'une manière plus directe que par (50), le choix des fonctions $\mu(\theta)$, $h(x)$, $l(x)$: *on peut prendre arbitrairement $h(x)$ et $l(x)$* [1]) *et alors $\mu(\theta)$ est déterminé par* (50) ou même mieux *par une formule explicite.* En effet, (50) peut s'écrire

$$e^{\Theta\mu' - \mu} = \int_{-\infty}^{+\infty} e^{\mu'\Theta h(x) + l(x)}\, dx.$$

Donnons-nous alors arbitrairement $h(x)$ et $l(x)$ et soit s une variable arbitraire : la fonction

$$\int_{-\infty}^{+\infty} e^{sh(x) + l(x)}\, dx$$ [1])

sera une fonction positive connue que nous pourrons représenter par $e^{\Psi(s)}$. On voit alors que $\mu(\theta)$ sera défini par

$$\theta\mu' - \mu = \psi(\mu') \quad \text{ou}$$

(55) $$\mu = \theta\mu' - \psi(\mu')$$

c'est-à-dire une équation de Clairaut. La solution μ' = constante réduirait $f(x, \theta)$, d'après (48) à une fonction indépendante de θ, cas où le problème n'aurait plus de sens. μ est donc donné par la solution singulière de (55), qui est unique et s'obtient en éliminant s entre $\mu = \theta s - \psi(s)$ et $\theta = \psi'(s)$ ou encore entre

$$e^{\Theta s - \mu} = \int_{-\infty}^{+\infty} e^{sh(x) + l(x)}\, dx \text{ et}$$

(55bis) $$\int_{-\infty}^{+\infty} e^{sh(x) + l(x)} [h(x) - \theta]\, dx = 0.$$

Si l'on veut, $\mu(\theta)$ est donné par la relation

$$e^{-\mu} = e^{-\Theta s} \int_{-\infty}^{+\infty} e^{sh(x) + l(x)}\, dx$$

où s est donné en fonction de θ par la relation implicite (55bis).

OPEN

Fisher Metric and Information Geometry (IG)

IG could be introduced with Koszul-Vinberg Characteristic Function:

$$\psi_\Omega(x) = \int_{\Omega^*} e^{-\langle x,\xi \rangle} d\xi, \ \forall x \in \Omega$$ with Ω and Ω^* are dual convex cones

$$\psi_\Omega(x+\lambda u) = \psi_\Omega(x) - \lambda \langle x^*, u \rangle + \frac{\lambda^2}{2} \langle K(x)u, u \rangle + \dots$$

Density is given by Solution of Maximum Entropy: $\Phi^*(\bar{\xi}) = -\int_{\Omega^*} p_{\bar{\xi}}(\xi) \log p_{\bar{\xi}}(\xi) d\xi$

$$\underset{p}{Max} \left[-\int_{\Omega^*} p_{\bar{\xi}}(\xi) \log p_{\bar{\xi}}(\xi) d\xi \right] \text{ such that } \int_{\Omega^*} p_{\bar{\xi}}(\xi) d\xi = 1 \text{ and } \int_{\Omega^*} \xi . p_{\bar{\xi}}(\xi) d\xi = \bar{\xi}$$

$$p_{\bar{\xi}}(\xi) = \frac{e^{-\langle \xi, \Theta^{-1}(\bar{\xi}) \rangle}}{\int_{\Omega^*} e^{-\langle \xi, \Theta^{-1}(\bar{\xi}) \rangle} d\xi} \text{ with } \bar{\xi} = \Theta(x) = \frac{\partial \Phi(x)}{\partial x} \text{ where } \Phi(x) = -\log \int_{\Omega^*} e^{-\langle x,\xi \rangle} d\xi = -\log \psi_\Omega(x)$$

The inversion $\Theta^{-1}(\bar{\xi})$ is given by Legendre transform based on :

$$\Phi^*(x^*) = \langle x, x^* \rangle - \Phi(x) \text{ with } x^* = \frac{d\Phi(x)}{dx} \quad \Phi(x) = -\log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi \quad \forall x \in \Omega \text{ and } \forall x^* \in \Omega^*$$

Fisher Metric and Information Geometry (IG)

- Maurice Fréchet, studying "distinguished functions" (densities with estimator reaching the Fréchet-Darmois bound), have also observed that solution should verify the Alexis Clairaut Equation:

$$\Phi^*(x^*) = \left\langle \Theta^{-1}(x^*), x^* \right\rangle - \Phi\left[\Theta^{-1}(x^*)\right] \quad \forall x^* \in \left\{\Theta(x) / x \in \Omega\right\}$$

- Fisher Metric appears as hessian of characteristic function logarithm:

$$\log p_x(\xi) = -\left\langle x, \xi \right\rangle + \Phi(x) \Rightarrow \frac{\partial^2 \log p_x(\xi)}{\partial x^2} = \frac{\partial^2 \Phi(x)}{\partial x^2}$$

$$I(x) = -E_\xi\left[\frac{\partial^2 \log p_x(\xi)}{\partial x^2}\right] = -\frac{\partial^2 \Phi(x)}{\partial x^2}$$

$$\frac{\partial^2 \Phi}{\partial x^2} = \left[\frac{\partial^2 \Phi^*}{\partial x^{*2}}\right]^{-1}$$

$$I(x) = \frac{\partial^2 \log \psi_\Omega(x)}{\partial x^2} = E_\xi\left[\xi^2\right] - E_\xi\left[\xi\right]^2 = Var(\xi)$$

Fisher Matrix, Cramer-Rao-Fréchet-Darmois Bound & Information Geometry

- Cramer-Rao –Fréchet-Darmois Bound has been introduced by Fréchet in 1939 and by Rao in 1945 as inverse of Fisher Matrix $I(\theta)$:

$$R_{\hat{\theta}} = E\left[\left(\theta - \hat{\theta}\right)\left(\theta - \hat{\theta}\right)^{+}\right] \leq I(\theta)^{-1} \qquad \left[I(\theta)\right]_{i,j} = -E\left[\frac{\partial^2 \log p_\theta(z)}{\partial\theta_i \partial\theta_j^*}\right]$$

- Rao has proposed to introduce a differential metric in space of parameters of probability density (axiomatized by N. Chentsov):

$$ds_\theta^2 = Kullback\_Divergence\left(p_\theta(z), p_{\theta+d\theta}(z)\right)$$

$$ds_\theta^2 = -\int p_\theta(z) \log \frac{p_{\theta+d\theta}(z)}{p_\theta(z)} dz$$

$$w = W(\theta) \Rightarrow ds_w^2 = ds_\theta^2$$

$$ds_\theta^2 \underset{Taylor}{\approx} \sum_{i,j} g_{ij} d\theta_i d\theta_j^* = \sum_{i,j} \left[I(\theta)\right]_{i,j} d\theta_i d\theta_j^* = d\theta^{+}.I(\theta).d\theta$$

Gibbs density (Maximum Entropy) and Legendre Transform

Maximum Entropy Principle for Density Estimation: Gibbs-Duhem Density

$$\underset{p}{Max}\left[-\int_{\Omega^*} p_{\hat{\xi}}(\xi)\log p_{\hat{\xi}}(\xi).d\xi\right] \text{ such that } \int_{\Omega^*} p_{\hat{\xi}}(\xi)d\xi = 1 \text{ and } \int_{\Omega^*} \xi.p_{\hat{\xi}}(\xi)d\xi = \hat{\xi}$$

$$p_{\hat{\xi}}(\xi) = \frac{e^{-\left\langle \Theta^{-1}(\hat{\xi}),\xi \right\rangle}}{\int_{\Omega^*} e^{-\left\langle \Theta^{-1}(\hat{\xi}),\xi\xi \right\rangle}.d\xi} \quad \hat{\xi} = \Theta(\beta) = \frac{\partial \Phi(\beta)}{\partial \beta} \text{ where } \Phi(\beta) = -\log \psi_\Omega(\beta)$$

$$\psi_\Omega(\beta) = \int_{\Omega^*} e^{-\langle \beta,\xi \rangle} d\xi \quad , \quad S(\hat{\xi}) = -\int_{\Omega^*} p_{\hat{\xi}}(\xi)\log p_{\hat{\xi}}(\xi).d\xi \text{ and } \beta = \Theta^{-1}(\hat{\xi})$$

$$S(\hat{\xi}) = \left\langle \hat{\xi}, \beta \right\rangle - \Phi(\beta)$$

LEGENDRE TRANSFORM

Fisher Metric and Information Geometry (IG)

Fisher Metric appears as hessian of characteristic function logarithm:

$$\log p_{\hat{\xi}}(\xi) = -\langle \xi, \beta \rangle + \Phi(\beta)$$

$$S(\hat{\xi}) = -\int_{\Omega^*} p_{\hat{\xi}}(\xi).\log p_{\hat{\xi}}(\xi).d\xi = -E\left[\log p_{\hat{\xi}}(\xi)\right]$$

$$S(\hat{\xi}) = \langle E[\xi], \beta \rangle - \Phi(\beta) = \langle \hat{\xi}, \beta \rangle - \Phi(\beta)$$

LEGENDRE TRANSFORM

$$I(\beta) = -E\left[\frac{\partial^2 \log p_\beta(\xi)}{\partial \beta^2}\right] = -E\left[\frac{\partial^2\left(-\langle \xi, \beta \rangle + \Phi(\beta)\right)}{\partial \beta^2}\right] = -\frac{\partial^2 \Phi(\beta)}{\partial \beta^2}$$

$$\hat{\xi} = \frac{\partial \Phi(\beta)}{\partial \beta}$$

$$I(\beta) = E\left[\frac{\partial \log p_\beta(\xi)}{\partial \beta} \frac{\partial \log p_\beta(\xi)}{\partial \beta}^T\right] = E\left[\left(\xi - \hat{\xi}\right)\left(\xi - \hat{\xi}\right)^T\right] = E\left[\xi^2\right] - E[\xi]^2 = Var(\xi)$$

2 metrics in dual coordinates systems for dual potential functions

1st Metric of Information Geometry: Fisher Metric = hessian of logarithm characteristic function

$$I(\beta) = -E\left[\frac{\partial^2 \log p_\beta(\xi)}{\partial \beta^2}\right] = -\frac{\partial^2 \Phi(\beta)}{\partial \beta^2}$$

$$ds_g^2 = d\beta^T I(\beta) d\beta = \sum_{ij} g_{ij} d\beta_i d\beta_j \quad \text{with} \quad g_{ij} = \left[I(\beta)\right]_{ij}$$

2nd Metric of Information Geometry: hessian of Shannon Entropy

$$\frac{\partial^2 S(\hat{\xi})}{\partial \hat{\xi}^2} = \left[\frac{\partial^2 \Phi(\beta)}{\partial \beta^2}\right]^{-1} \qquad S(\hat{\xi}) = \left\langle \hat{\xi}, \beta \right\rangle - \Phi(\beta)$$

$$ds_h^2 = d\hat{\xi}^T \left[\frac{\partial^2 S(\hat{\xi})}{\partial \hat{\xi}^2}\right] d\hat{\xi} = \sum_{ij} h_{ij} d\hat{\xi}_i d\hat{\xi}_j \quad \text{with} \quad h_{ij} = \left[\frac{\partial^2 S(\hat{\xi})}{\partial \hat{\xi}^2}\right]_{ij}$$

Same Distance for Dual metrics

$$ds_g^2 = ds_h^2$$

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THALES

Example : Gaussian scalar distribution

For Gaussian Law, Fisher Information matrix is given by :

$$I(\theta) = \sigma^{-2} . \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{with} \quad E\left[(\theta - \hat{\theta})(\theta - \hat{\theta})^T \right] \geq I(\theta)^{-1} \quad \text{and} \quad \theta = \begin{pmatrix} m \\ \sigma \end{pmatrix}$$

- Fisher matrix induced the following differential metric :

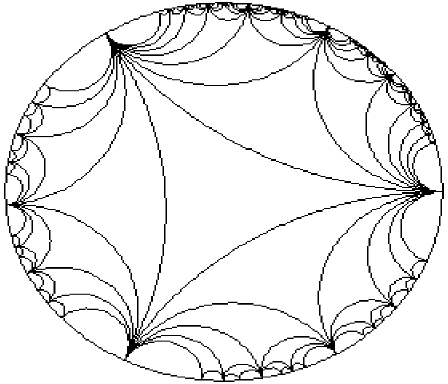
$$ds^2 = d\theta^T . I(\theta) . d\theta = \frac{dm^2}{\sigma^2} + 2.\frac{d\sigma^2}{\sigma^2} = 2.\sigma^{-2}\left[\left(\frac{dm}{\sqrt{2}} \right)^2 + (d\sigma)^2 \right]$$

- Poincaré model of hyperbolic Space :

$$z = \frac{m}{\sqrt{2}} + i.\sigma \qquad \omega = \frac{z - i}{z + i} \quad (|\omega| < 1)$$

$$\Rightarrow ds^2 = 8.\frac{|d\omega|^2}{\left(1 - |\omega|^2\right)^2}$$

Geometry of Gaussian Law Is Geometry of Hyperbolic Poincare Space



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THALES

Example : Gaussian scalar distribution

Gaussian Law Metric :

- If we set $r=|\omega|$, we can integrate along one radial :

$$ds^2=8.\left(\frac{dr}{1-r^2}\right)^2=2.d\ln\frac{1+r}{1-r}$$

- Homeomorphisme is used then to compute distance between two arbitrary points in the unit disk :

$$v=\phi_\tau(\omega)=\frac{\omega-\tau}{\bar{\tau}\omega-1}.e^{j.\varphi}\quad\text{and}\quad 0=\phi_\tau(\tau)$$

- Distance between two Gaussian Law is then given by :

$$D^2(\{m_1,\sigma_1\},\{m_2,\sigma_2\})=2.\left(\ln\frac{1+\delta(\omega,\tau)}{1-\delta(\omega,\tau)}\right)^2\quad\text{with}\quad\delta(\omega,\tau)=\left|\frac{\omega-\tau}{1-\omega\bar{\tau}}\right|$$

$$z=\frac{m}{\sqrt{2}}+i.\sigma\quad\text{and}\quad\omega=\frac{z-i}{z+i}$$

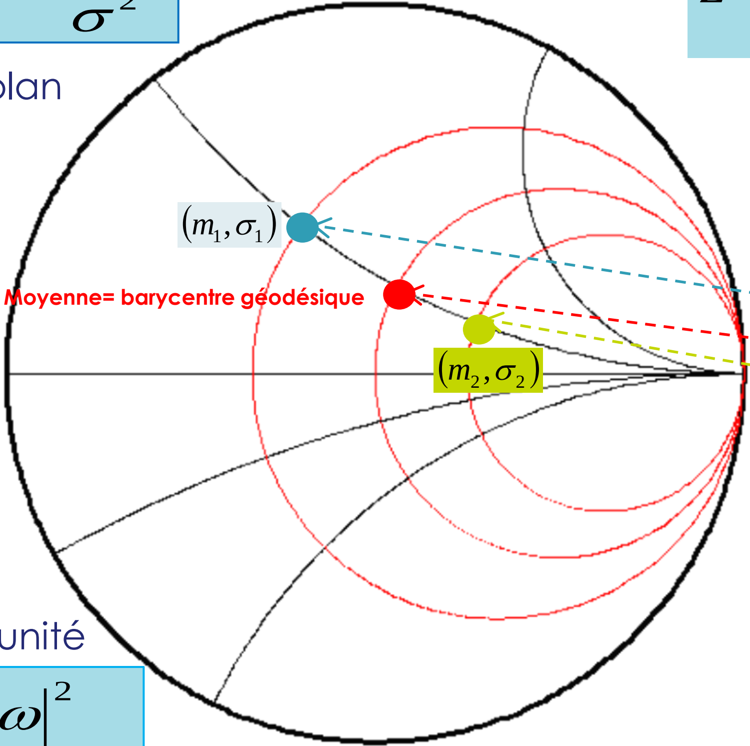
Monovariate Gaussian = 1 point in Poincaré Unit Disk

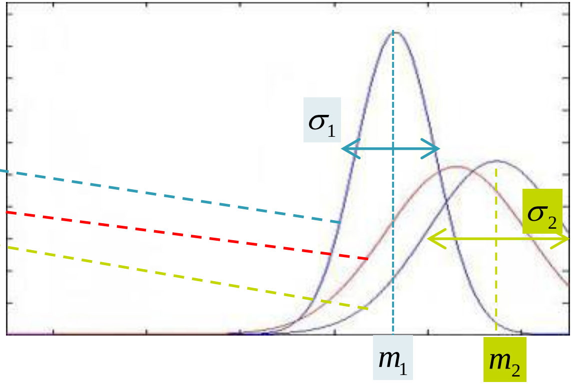
$$ds^2 = \frac{dm^2}{\sigma^2} + 2.\frac{d\sigma^2}{\sigma^2}$$

Métrique demi-plan

$$z = \frac{m}{\sqrt{2}} + i.\sigma$$

$$\omega = \frac{z - i}{z + i} \quad (|\omega| < 1)$$





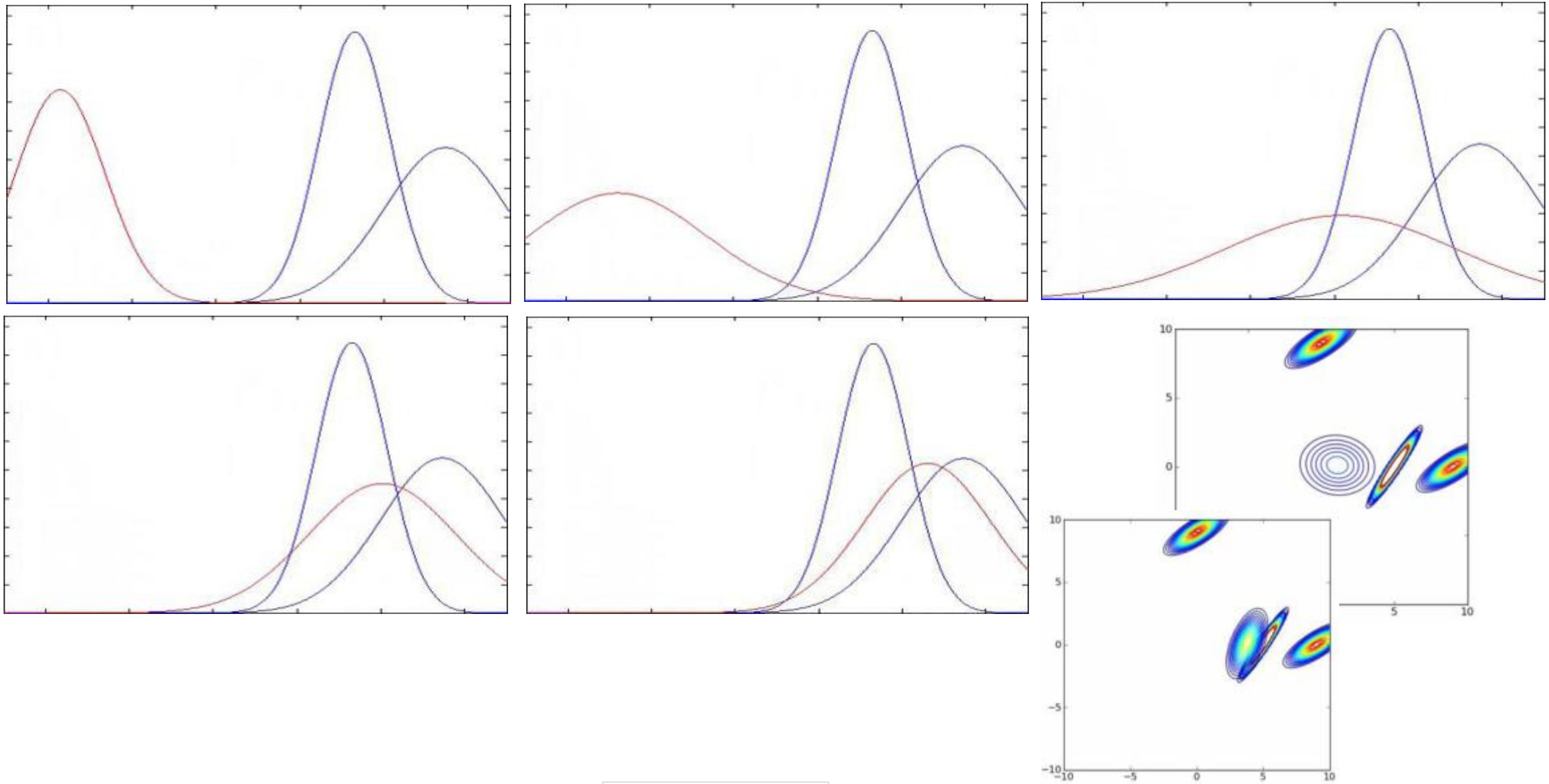
Métrique disque unité

$$ds^2 = 8.\frac{|d\omega|^2}{\left(1 - |\omega|^2\right)^2}$$

$$d^2(\{m_1,\sigma_1\},\{m_2,\sigma_2\}) = 2.\left(\log \frac{1+\delta(\omega^{(1)},\omega^{(2)})}{1-\delta(\omega^{(1)},\omega^{(2)})}\right)^2$$

avec $\delta(\omega^{(1)},\omega^{(2)}) = \left|\frac{\omega^{(1)} - \omega^{(2)}}{1 - \omega^{(1)}\omega^{(2)*}}\right|$

Mean/Mediane of Gaussian densities



THALES

Example of Multivariate Gaussian Law (real case)

Multivariate Gaussian law parameterized by moments

$$p_{\hat{\xi}}(\xi) = \frac{1}{(2\pi)^{n/2}\det(R)^{1/2}} e^{-\frac{1}{2}(z-m)^T R^{-1}(z-m)}$$

$$\frac{1}{2}(z-m)^T R^{-1}(z-m) = \frac{1}{2}\left[z^T R^{-1} z - m^T R^{-1} z - z^T R^{-1} m + m^T R^{-1} m\right]$$

$$= \frac{1}{2} z^T R^{-1} z - m^T R^{-1} z + \frac{1}{2} m^T R^{-1} m$$

$$p_{\hat{\xi}}(\xi) = \frac{1}{(2\pi)^{n/2}\det(R)^{1/2} e^{\frac{1}{2}m^T R^{-1} m}} e^{-\left[-m^T R^{-1} z + \frac{1}{2} z^T R^{-1} z\right]} = \frac{1}{Z} e^{-\langle \xi, \beta \rangle}$$

$$\xi = \begin{bmatrix} z \\ zz^T \end{bmatrix} \text{ and } \beta = \begin{bmatrix} -R^{-1}m \\ \frac{1}{2}R^{-1} \end{bmatrix} = \begin{bmatrix} a \\ H \end{bmatrix} \text{ with } \langle \xi, \beta \rangle = a^T z + z^T H z = Tr\left[z a^T + H^T z z^T\right]$$

Example of Multivariate Gaussian Law (real case)

Multivariate Gaussian Density given by their moments (and not cumulants)

$$p_{\hat{\xi}}(\xi) = \frac{1}{\int\limits_{\Omega^*} e^{-\langle \xi,\beta \rangle}.d\xi} e^{-\langle \xi,\beta \rangle} = \frac{1}{Z} e^{-\langle \xi,\beta \rangle} \quad \text{with } \log(Z) = n\log(2\pi) + \frac{1}{2}\log\det(R) + \frac{1}{2} m^T R^{-1} m$$

$$\xi = \begin{bmatrix} z \\ zz^T \end{bmatrix}, \hat{\xi} = \begin{bmatrix} E[z] \\ E[zz^T] \end{bmatrix} = \begin{bmatrix} m \\ R + mm^T \end{bmatrix}, \beta = \begin{bmatrix} a \\ H \end{bmatrix} = \begin{bmatrix} -R^{-1}m \\ \frac{1}{2}R^{-1} \end{bmatrix} \quad \text{with } \langle \xi,\beta \rangle = Tr\left[za^T + H^T zz^T\right]$$

$$R = E\left[(z-m)(z-m)^T\right] = E\left[zz^T - mz^T - zm^T + mm^T\right] = E\left[zz^T\right] - mm^T$$

1st Potential function (Free Energy / logarithm of characteristic function)

$$\psi_\Omega(\beta) = \int\limits_{\Omega^*} e^{-\langle \xi,\beta \rangle}.d\xi \quad \text{and} \quad \Phi(\beta) = -\log\psi_\Omega(\beta) = \frac{1}{2}\left[-Tr\left[H^{-1}aa^T\right] + \log\left[(2)^n \det H\right] - n\log(2\pi)\right]$$

Relation between 1st Potential function and moment

$$\frac{\partial\Phi(\beta)}{\partial\beta} = \frac{\partial\left[-\log\psi_\Omega(\beta)\right]}{\partial\beta} = \int\limits_{\Omega^*} \xi \frac{e^{-\langle \xi,\beta \rangle}}{\int\limits_{\Omega^*} e^{-\langle \xi,\beta \rangle}.d\xi}.d\xi = \int\limits_{\Omega^*} \xi.p_{\hat{\xi}}(\xi)d\xi = \hat{\xi}$$

THALES

Example of Multivariate Gaussian Law (real case)

2nd Potential function (Shannon Entropy) as Legendre Transform of 1st one:

$$S(\hat{\xi})=\left\langle \hat{\xi},\beta \right\rangle - \Phi(\beta) \quad \text{with} \quad \frac{\partial \Phi(\beta)}{\partial \beta}=\hat{\xi} \quad \text{and} \quad \frac{\partial S(\hat{\xi})}{\partial \hat{\xi}}=\beta$$

$$S\left(\hat{\xi}\right)=-\int_{\Omega^*}\frac{e^{-\langle \xi,\beta\rangle}}{\int_{\Omega^*} e^{-\langle \xi,\beta\rangle}.d\xi}\log\frac{e^{-\langle \xi,\beta\rangle}}{\int_{\Omega^*} e^{-\langle \xi,\beta\rangle}.d\xi}.d\xi=-\int_{\Omega^*} p_{\hat{\xi}}(\xi)\log p_{\hat{\xi}}(\xi).d\xi$$

How to make Density dependent on moments only:

$$\hat{\xi}=\frac{\partial \Phi(\beta)}{\partial \beta}=\Theta(\beta)\Rightarrow \beta=\Theta^{-1}(\hat{\xi}) \quad \text{or} \quad \beta=\frac{\partial S(\hat{\xi})}{\partial \hat{\xi}}\Rightarrow \beta=\begin{bmatrix} a \\ H \end{bmatrix}=\begin{bmatrix} -R^{-1}m \\ \frac{1}{2}R^{-1} \end{bmatrix}$$

$$p_{\hat{\xi}}(\xi)=\frac{1}{\int_{\Omega^*} e^{-\langle \xi,\beta\rangle}.d\xi}e^{-\langle \xi,\beta\rangle} \quad \text{with} \quad \langle \xi,\beta\rangle=a^T z+z^T Hz=Tr\left[za^T+H^T zz^T\right]=-m^T R^{-1}z+\frac{1}{2}z^T R^{-1}z$$

$$S(\hat{\xi})=-\int_{\Omega^*} p_{\hat{\xi}}(\xi)\log p_{\hat{\xi}}(\xi).d\xi=\frac{1}{2}\left[\log(2)^n\det\left[H^{-1}\right]+n\log(2\pi.e)\right]=\frac{1}{2}\left[\log\det[R]+n\log(2\pi.e)\right]$$

General Scheme based on Cartan-Killing form

$\langle .,. \rangle$ inner product from Cartan - Killing Form :

$$\langle \hat{\xi}, \beta \rangle = -B\left(\hat{\xi}, \theta(\beta)\right) \quad \text{with} \quad B\left(\hat{\xi}, \theta(\beta)\right) = Tr\left(Ad_{\hat{\xi}} Ad_{\theta(\beta)}\right)$$

$$S(\hat{\xi}) = \langle \hat{\xi}, \beta \rangle - \Phi(\beta)$$

$$S(\hat{\xi}) = -\int_{\Omega^*} p_{\hat{\xi}}(\xi) \log p_{\hat{\xi}}(\xi).d\xi$$

Legendre Transform

$$\Phi(\beta) = -\log \psi_\Omega(\beta)$$

$$\text{with} \quad \psi_\Omega(\beta) = \int_{\Omega^*} e^{-\langle \beta, \xi \rangle} d\xi$$

$$p_{\hat{\xi}}(\xi) = \frac{e^{-\langle \Theta^{-1}(\hat{\xi}), \xi \rangle}}{\int_{\Omega^*} e^{-\langle \Theta^{-1}(\hat{\xi}), \xi\xi \rangle}.d\xi} \qquad \hat{\xi} = \Theta(\beta) = \frac{\partial \Phi(\beta)}{\partial \beta}$$

$$\beta = \frac{\partial S(\hat{\xi})}{\partial \hat{\xi}}$$

$$I(\beta) = -E\left[\frac{\partial^2 \log p_\beta(\xi)}{\partial \beta^2}\right]$$

$$I(\beta) = -\frac{\partial^2 \Phi(\beta)}{\partial \beta^2}$$

$$ds_g^2 = \sum_{ij} g_{ij} d\beta_i d\beta_j \quad \text{with} \quad g_{ij} = \left[\frac{\partial^2 \Phi(\beta)}{\partial \beta^2}\right]_{ij}$$

$$ds_g^2 = ds_h^2$$

$$ds_h^2 = \sum_{ij} h_{ij} d\hat{\xi}_i d\hat{\xi}_j \quad \text{with} \quad h_{ij} = \left[\frac{\partial^2 S(\hat{\xi})}{\partial \hat{\xi}^2}\right]_{ij}$$

Application for Density of Symmetric Positive Definite Matrices

If we apply previous equation for Symmetric Positive Definite Matrices:

$$p_{\hat{\xi}}(\xi) = \frac{e^{-\left\langle \Theta^{-1}(\hat{\xi}),\xi \right\rangle}}{\int_{\Omega^*} e^{-\left\langle \Theta^{-1}(\hat{\xi}),\xi\xi \right\rangle}.d\xi} \quad \hat{\xi} = \Theta(\beta) = \frac{\partial \Phi(\beta)}{\partial \beta}$$

$$\Phi(\beta) = -\log \psi_\Omega(\beta)$$

$$\text{with} \quad \psi_\Omega(\beta) = \int_{\Omega^*} e^{-\langle \beta,\xi \rangle} d\xi$$

$$\langle \eta, \xi \rangle = Tr\left(\eta^T \xi\right), \quad \forall \eta, \xi \in Sym(n)$$

Application:

$$\psi_\Omega(\beta) = \int_{\Omega^*} e^{-\langle \beta,\xi \rangle} d\xi = \det(\beta)^{-\frac{n+1}{2}} \psi_\Omega(I_d)$$

$$\hat{\xi} = \frac{\partial \Phi(\beta)}{\partial \beta} = \frac{\partial(-\log \psi_\Omega(\beta))}{\partial \beta} = \frac{n+1}{2}\beta^{-1}$$

$$p_{\hat{\xi}}(\xi) = e^{-\left\langle \Theta^{-1}(\hat{\xi}),\xi \right\rangle + \Phi\left(\Theta^{-1}(\hat{\xi})\right)} = \psi_\Omega(I_d).\left[\det\left(\alpha \hat{\xi}^{-1}\right)\right] e^{-Tr\left(\alpha \hat{\xi}^{-1} \xi\right)} \quad \text{with} \quad \alpha = \frac{n+1}{2}$$

Fisher Metric and Euler-Lagrange Equation

Fisher Metric for Multivariate Gaussian Law

$$ds^2 = \sum_{ij} g_{ij} d\theta_i d\theta_j = dm^T R^{-1} dm + \frac{1}{2} Tr\left[\left(R^{-1} dR\right)^2\right]$$

Classical Euler-Lagrange equation

$$\sum_{i=1}^{n} g_{ik} \ddot{\theta}_i + \sum_{i,j=1}^{n} \Gamma_{ijk} \dot{\theta}_i \dot{\theta}_j = 0 \quad , \quad k = 1, \ldots, n$$

$$\text{with} \quad \Gamma_{ijk} = \frac{1}{2}\left[\frac{\partial g_{jk}}{\partial \theta_i} + \frac{\partial g_{jk}}{\partial \theta_j} + \frac{\partial g_{ij}}{\partial \theta_k}\right]$$

$$\Rightarrow \begin{cases} \ddot{R} + \dot{m}\dot{m}^T - \dot{R}R^{-1}\dot{R} = 0 \\ \ddot{m} - \dot{R}R^{-1}\dot{m} = 0 \end{cases}$$

We cannot easily integrate this Euler-Lagrange Equation (we will see that Lie group Theory will provide new equation: Euler-Poincaré equation)

Souriau Chapter on Statistical Physics: Multivariate Gaussian Law

Exemple : (*loi normale*) :

Prenons le cas $V = R^n$, λ = mesure de Lebesgue, $\Psi(x) \equiv \begin{pmatrix} x \\ x \otimes x \end{pmatrix}$; un élément Z du dual de E peut se définir par la formule

$$Z(\Psi(x)) \equiv \overline{a}.x + \tfrac{1}{2}\overline{x}.H.x$$

[$a \in R^n$; H = matrice symétrique]. On vérifie que la convergence de l'intégrale I_0 a lieu si la matrice H est positive (1) ; dans ce cas la loi de Gibbs s'appelle *loi normale de Gauss* ; on calcule facilement I_0 en faisant le changement de variable $x^* = H^{1/2}x + H^{-1/2}a$ (2) ; il vient

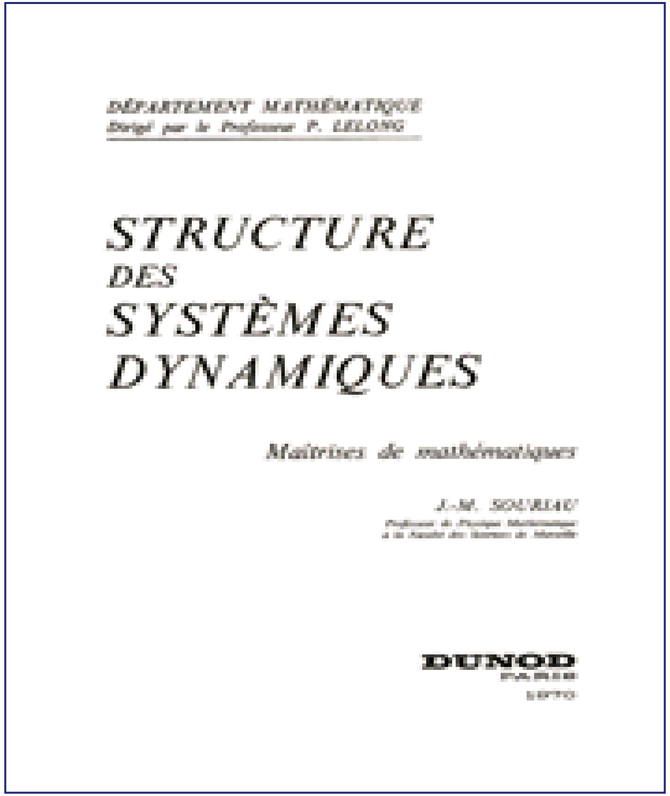
$$z = \tfrac{1}{2}\left[\overline{a}.H^{-1}.a - \log(\text{dét}(H)) + n\log(2\pi)\right]$$

alors la convergence de I_1 a lieu également ; on peut donc calculer M, qui est défini par les moments du premier et du second ordre de la loi (16.196) ; le calcul montre que le moment du premier ordre est égal à $-H^{-1}.a$ et que les composantes du tenseur *variance* (16.196) sont égales aux éléments de la matrice H^{-1} ; le moment du second ordre s'en déduit immédiatement.

La formule (16.200) donne l'*entropie* :

$$s = \frac{n}{2}\log(2\pi e) - \frac{1}{2}\log(\text{dét}(H))$$;.

(1) Voir *Calcul linéaire*, tome II.
(2) C'est-à-dire en recherchant l'*image* de la loi par l'application $x \mapsto x^*$.


DÉPARTEMENT MATHÉMATIQUE
Dirigé par le Professeur P. LELONG

STRUCTURE DES SYSTÈMES DYNAMIQUES

Maîtrises de mathématiques

J.-M. SOURIAU

DUNOD

http://www.jmsouriau.com/structure_des_systemes_dynamiques.htm

Maximum Entropy / Gibbs Density for Multivariate Gaussian Law

- if we take vector with tensor components $\xi = \begin{pmatrix} z \\ z \otimes z \end{pmatrix}$, components of $\bar{\xi}$ will provide moments of 1st and 2nd order of the density of probability $p_{\bar{\xi}}(\xi)$, that is defined by Gaussian law. In this particular case, we can write:

$$\langle \xi, x \rangle = a^T z + \frac{1}{2} z^T H z \quad \text{with} \quad a \in R^n \quad \text{and} \quad H \in Sym(n)$$

- By change of variable given by $z' = H^{1/2} z + H^{-1/2} a$, we can then compute the logarithm of the Koszul characteristic function:

$$\Phi(x) = -\frac{1}{2}\left[a^T H^{-1} a + \log\det\left[H^{-1}\right] + n \log(2\pi)\right]$$

- We can prove that 1st moment is equal to $-H^{-1}a$ and that components of variance tensor are equal to elements of matrix H^{-1}, that induces the 2nd moment. The Koszul Entropy, its Legendre transform, is then given by:

$$\Phi^*(\bar{\xi}) == \frac{1}{2}\left[\log\det\left[H^{-1}\right] + n \log(2\pi.e)\right]$$

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J. M. SOURIAU

DYNAMIC SYSTEMS STRUCTURE

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TRANSFORMATION DE LAPLACE.

Définition, théorème:

Soit E un espace vectoriel de dimension finie, μ une mesure de son dual E^*.

Alors la fonction définie par

$$\Theta \mapsto \int_E e^{M\Theta} \mu(M)\, dM \qquad [M \in E^*]$$

(?.17 b) pour tout $\Theta \in E$ tel que l'intégrale soit convergente s'appelle transformée de Laplace de μ ; son ensemble de définition est un convexe de E.

Soient $\Theta_1, \Theta_2 \in \mathrm{def}(F)$, F désignant cette transformée de Laplace;
Soit $s \in [0,1]$, $\Theta = [1-s]\Theta_1 + s\,\Theta_2$. Les fonctions $f_1 : M \mapsto e^{M\Theta_1}$, f_2 : $M \mapsto e^{M\Theta_2}$ sont μ-intégrables; il faut montrer que $f : M \mapsto e^{M\Theta}$ est aussi μ-intégrable. Or la fonction $\varphi = [1-s]\, f_1 + s\, f_2$ est μ-intégrable (17.106), donc $|\mu|$-intégrable (17.110); comme $x \mapsto e^x$ est convexe (16.6?) (16.47), on a $e^{M\Theta} = e^{[1-s]\Theta_1 + s\Theta_2} \leq [1-s]\, e^{M\Theta_1} + s\, e^{M\Theta_2}$; donc $|f| \leq \varphi$;
f est μ-intégrable (17.109).

C.Q.F.D.

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Théorème (notations(17.17)):

La transformée de Laplace F de la mesure μ est différentiable dans l'intérieur de son ensemble de définition def(F); sa dérivée p-ème est donnée par l'intégrale convergente

(17.182)
$$D^p(F)(\Theta) = \int_{E^*} M \otimes M \otimes \dots \otimes M \; e^{M.\Theta} \; \mu(M)\, dM$$

en tout point intérieur à def(F).

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Théorème:

(17.185) Soit E un espace vectoriel de dimension finie, μ une mesure positive non nulle du dual E^*, F sa transformée de Laplace (17.17?). Alors :

1°) F est une fonction convexe semi-continue; $F(\Theta) > 0 \quad \forall \Theta \in \mathrm{def}(F)$;

2°) $f = \mathrm{Log} \circ F$ est convexe et semi-continue ;

3°) Soit Θ un point intérieur de $\mathrm{def}(F)$. Alors :

a) $D^2(f)(\Theta) \geq 0$;

b) $$D^2(f)(\Theta) = \int_{E^*} e^{M.\Theta} \left[M - D(f)(\Theta)\right]^{\otimes 2} \mu(M) \, dM$$

c) $\left[D^2(f)(\Theta) \text{ inversible}\right] \Leftrightarrow \left[\text{enveloppe affine}(\text{support}(\mu)) = E^*\right]$

Vérification laissée au lecteur (quelques indications: pour montrer le 2°, vérifier que l'épigraphe de f est convexe et fermé; pour vérifier 3°b, considérer les vecteurs V qui font éventuellement partie du noyau de $D^2(f)(\Theta)$)

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ENTROPIE

Lemme :

(17.190) Soit X un espace localement compact; soit λ une mesure positive de X ayant X comme support.

Alors la fonction Φ :

$$\Phi(h) = \text{Log} \int_X e^{h(x)} \lambda(x)\, dx \quad \left[\forall h \in \mathcal{C}(X) \text{ tel que l'intégrale converge} \right.$$

est convexe.

D'après (17.141), l'intégrale est strictement positive lorsqu'elle est convergente, ce qui assure l'existence de son logarithme. L'épigraphe de Φ (16.46) est l'ensemble des $\binom{h}{y}$ tels que $\int_X e^{h(x)-y} \lambda(x)\, dx \leqslant 1$; la convexité de l'exponentielle montre que cet épigraphe est convexe.

C.Q.F.D.

Nous allons définir – sous le nom de négentropie – une transformée de Legendre formelle de cette fonction Φ :

Définition (suite de (17.190))

(17.191) Nous appellerons loi de Boltzmann (relative à λ) toute mesure μ de X telle que l'ensemble de réels

$$\mu(h) - \Phi(h) \quad \left[\begin{matrix} h \in \text{def}(\Phi) \\ \text{et} \\ h \quad \mu\text{-intégrable} \end{matrix} \right]$$

SOURIAU MODEL OF INFORMATION GEOMETRY: Lie Group Thermodynamics

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Covariant Gibbs Equilibrium

- Jean-Marie Souriau has observed in 1966 in « *Définition covariante des équilibres thermodynamiques* » that **Classical Gibbs Equilibrium is not covariant with respect to Dynamic Groups** (Gallilee Group in classical Mechanic or Poincaré Group in Relativity). Classical thermodynamics corresponds to the case of Time translation.

- To solve this incoherency, Souriau has extended definition of Canonical Gibbs Ensemble to Symplectic Manifolds on which a Lie Group has a Symplection Action:
  - (Planck) Temperature is an element of the Dynamic Group Lie Algebra
  - Heat is an element of the Dynamic Group Dual Lie Algebra

- In case of non-commutative groups, specific properties appear: the symmetry is spontaneously broken, some cohomological type of relationships are satisfied in the algebra of the Lie group

Gibbs Equilibrium: Solution of Maximum Entropy

- Let M be a differentiable manifold with a continuous positive density $d\omega$ and let E a finite vector space and $U(\xi)$ a continuous function defined on M with values in E. A continuous positive function $p(\xi)$ solution of this problem with respect to calculus of variations:

$$\underset{p(\xi)}{ArgMin}\left[s=-\int_M p(\xi)\log p(\xi)d\omega \right] \text{ such that } \begin{cases} \int_M p(\xi)d\omega=1 \\ \int_M U(\xi)p(\xi)d\omega=Q \end{cases}$$

- Solution: $p(\xi)=e^{\Phi(\beta)-\langle\beta,U(\xi)\rangle}$ with $Q=\dfrac{\int_M U(\xi)e^{-\langle\beta,U(\xi)\rangle}d\omega}{\int_M e^{-\langle\beta,U(\xi)\rangle}d\omega}$ and $\Phi(\beta)=-\log\int_M e^{-\langle\beta,U(\xi)\rangle}d\omega$

- Entropy $s=-\int_M p(\xi)\log p(\xi)d\omega$ can be stationary only if there exist a scalar Φ and an element β belonging to the dual of E.

- Entropy appears naturally as Legendre transform of Φ :

$$s(Q)=\langle\beta,Q\rangle-\Phi(\beta)$$

THALES

Gibbs Canonical Ensemble

- This value $s(Q) = \langle \beta, Q \rangle - \Phi(\beta)$ is a strict minimum of s, and the equation:

$$Q = \frac{\int_M U(\xi) e^{-\langle \beta, U(\xi) \rangle} d\omega}{\int_M e^{-\langle \beta, U(\xi) \rangle} d\omega}$$

has a maximum of one solution for each value of Q.

- The function $\Phi(\beta)$ is differentiable and we can write $d\Phi = d\beta.Q$

and identifying *E* with its dual: $Q = \frac{\partial \Phi}{\partial \beta}$

- Uniform convergence of $\int_M U(\xi) \otimes U(\xi) e^{-\langle \beta, U(\xi) \rangle} d\omega$ proves that $-\frac{\partial^2 \Phi}{\partial \beta^2} > 0$ and that $-\Phi(\beta)$ is convex.

- Then, $Q(\beta)$ and $\beta(Q)$ are mutually inverse and differentiable, with $ds = \beta.dQ$
- Identifying E with its bidual: $\beta = \frac{\partial s}{\partial Q}$

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Gibbs Canonical Ensemble on Symplectic Manifold

- In statistical mechanics, a canonical ensemble is the statistical ensemble that is used to represent the possible states of a mechanical system that is being maintained in thermodynamic equilibrium.

- Souriau has extended this notion of **Gibbs canonical ensemble on Symplectic manifold M for a Lie group action on *M***

- The seminal idea of Lagrange was to consider that a **statistical state is simply a probability measure on the manifold of motions**

- In Jean-Marie Souriau approach, one movement of a dynamical system (classical state) is a point on manifold of movements.

- For statistical mechanics, the movement variable is replaced by a random variable where a statistical state is probability law on this manifold.

Gibbs Canonical Ensemble on Symplectic Manifold

- In classical statistical mechanics, a state is given by the solution of **Liouville equation** on the phase space, the partition function.

- As symplectic manifolds have a completely continuous measure, invariant by diffeomorphisms, the **Liouville measure** λ , all statistical states will be the product of Liouville measure by the scalar function given by the generalized partition function $e^{\Phi(\beta)-\langle\beta,U(\xi)\rangle}$ defined by:
  - the energy U (defined in dual of Lie Algebra of the dynamic group)
  - the geometric temperature β
  - $\Phi(\beta)$ a normalizing constant such the mass of probability is equal to 1

- The Gibbs equilibrium state is extended to all Symplectic manifolds with a dynamic group. To ensure that all integrals could converge, the canonical Gibbs ensemble is the largest open proper subset (in Lie algebra) where these integrals are convergent. This canonical Gibbs ensemble is convex.

Gibbs Ensemble of a Dynamic Group

Let *M* a Symplectic Manifold with a Dynamic Group G:

- G is a Lie Group that acts on M by « Symplectomorphisms » $\Psi_g : G \times M \to M$
- Let Ψ_g transformation of M by the element $g \in G$: $g \times x \mapsto \Psi_g(x)$
- Let $Z \in \mathfrak{g}$ **an element of Lie Algebra** $\mathfrak{g}$ of Dynamic Group G , we can associate **a vectors Field** Z_M defined on M , that characterize infinitesimal action of G

$$\frac{\partial}{\partial t}\left[\exp(tZ)_M(x)\right] = Z_M\left(\exp(tZ)_M(x)\right)$$

- Dynamic Group G has a **moment map**, a differentiable application from the manifold M to $\mathfrak{g}^*$ the dual of Lie Algebra $\mathfrak{g}$: $x \in M \mapsto U \in \mathfrak{g}^*$.
- The moment map is charcaterized by the equation:

$$\sigma\left(\delta x, Z_M(x)\right) = \delta U(Z)$$

  - σ Symplectic form of M
  - δx arbitrary variation of point x on M
  - δU associate variation of U to δx

Geometric (Planck) Temperature in the Lie Algrbra

- Let a Group G of a Manifold M with a moment map E, the **Geometric (Planck) Temperature** β is all elements of Lie Agebra $\mathfrak{g}$ of G such that the following integrals converges in a neighborhood of β : $I_0(\beta)=\int_M e^{-\langle \beta,U\rangle}d\lambda$
  - $\langle \beta,U\rangle$ notes the duality of $\mathfrak{g}$ and $\mathfrak{g}^*$
  - $d\lambda$ is the Liouville density on M

- Theorem: The function I_0 is infinitly differentiable C^∞ in Ω (the largest open proper subset of $\mathfrak{g}$) and is n[th] derivative for all $\beta \in \Omega$, the tensor integral is convergent: $I_n(\beta)=\int_M e^{-\langle \beta,U\rangle}U^{\otimes n}d\lambda$

- To each temperature β, we can associate probability law on M with distribution function (such that the probability law has a mass equal to 1):

$$e^{\Phi(\beta)-\langle \beta,U(\xi)\rangle} \text{ with } \Phi(\beta)=-\log(I_0)=-\log\int_M e^{-\langle \beta,U(\xi)\rangle}d\lambda \text{ and } Q(\beta)=\int_M e^{\Phi(\beta)-\langle \beta,U(\xi)\rangle}Ud\lambda=\frac{I_1}{I_0}$$

  - The set of these probalities law is **Gibbs Ensemble of the Dynamic Group**, Φ is the **Thermodynamic Potential** and Q is the **Geometric Heat** $Q\in\mathfrak{g}^*$

THALES

Geometric Fisher Metric: Geometric Heat Capacity

- We can observe that the Geometric Heat Q is C^{∞} function of Geometric Temperature β in Dual Lie Algebra $\mathfrak{g}^*$:

$$\beta \in \mathfrak{g} \mapsto Q \in \mathfrak{g}^*$$

$$Q(\beta) = \int_M e^{\Phi(\beta) - \langle \beta, U(\xi) \rangle} U d\lambda = \frac{I_1}{I_0}$$

- We have: $Q = \dfrac{\partial \Phi}{\partial \beta}$

- Its derivative is a 2nd order symmetric tensor: $\dfrac{\partial Q}{\partial \beta} = \dfrac{I_2}{I_0} - \dfrac{I_1 \otimes I_1}{I_0} = \dfrac{I_2}{I_1} - Q \otimes Q$

$$-\frac{\partial Q}{\partial \beta} = \int_M e^{\Phi(\beta) - \langle \beta, U(\xi) \rangle} [U - Q] \otimes [U - Q] d\lambda$$

$$-\frac{\partial Q}{\partial \beta} \geq 0$$

$$-\frac{\partial Q}{\partial \beta} = -\frac{\partial^2 \Phi}{\partial \beta^2}$$

- This quatratic form is positive, and positive definite for each $x \in M$ unless there exist a non null element $Z \in \mathfrak{g}$ such that $\langle U - Q, Z \rangle = 0$ (means that the moment U varies in an affine sub-manifold of $\mathfrak{g}^*$)

THALES

Geometric EntropyLegendre Transform

- **We have the inequality:**

$$\langle \beta_1 - \beta_0, Q_0 \rangle \geq \Phi_1 - \Phi_0$$

- **The application $\beta \mapsto Q$ is injective and as is derivative $\frac{\partial Q}{\partial \beta}$ is invertible, then this application $\beta \mapsto Q$ is a diffeomorphism of Ω on the open $\Omega^* \in \mathfrak{g}^*$**

- **We can then apply the Legendre Transform:** $Q \mapsto \left\langle \frac{\partial \Phi}{\partial \beta}, \beta \right\rangle - \Phi$

from which we obtain the Shannon Entropy:

$$\left\langle \frac{\partial \Phi}{\partial \beta}, \beta \right\rangle - \Phi = -\int_M e^{\Phi(\beta) - \langle \beta, U(\xi) \rangle} \left[\Phi(\beta) - \langle \beta, U(\xi) \rangle \right] d\lambda = -\int_M p \log p . d\lambda = s(Q)$$

with $p = e^{\Phi(\beta) - \langle \beta, U(\xi) \rangle}$

Legendre Transform

We have the reciprocal formula:

$$Q = \frac{\partial \Phi}{\partial \beta} \qquad \beta = \frac{\partial s}{\partial Q}$$

$$s(Q) = \left\langle \frac{\partial \Phi}{\partial \beta}, \beta \right\rangle - \Phi \qquad \Phi(\beta) = \left\langle Q, \frac{\partial s}{\partial Q} \right\rangle - s$$

For Classical Thermodynamics (Time translation only), we recover the definition of Boltzmann Entropy:

$$\begin{cases} \beta = \dfrac{\partial s}{\partial Q} \\ \beta = \dfrac{1}{T} \end{cases} \Rightarrow ds = \frac{dQ}{T}$$

Example of Gallileo Group

- The Galileo group of an observer is the group of affine maps

$$\begin{cases} \vec{x}' = R.\vec{x} + \vec{u}.t + \vec{w} \\ t' = t + e \end{cases}$$

$$\vec{x}, \vec{u} \text{ and } \vec{w} \in R^3, e \in R^+$$

$$R \in SO(3)$$

- Matrix Form of Gallileo Group

$$\begin{bmatrix} \vec{x}' \\ t' \\ 1 \end{bmatrix} = \begin{bmatrix} R & \vec{u} & \vec{w} \\ 0 & 1 & e \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{x} \\ t \\ 1 \end{bmatrix}$$

- Symplectic cocycles of the Galilean group: V. Bargmann (Ann. Math. 59, 1954, pp 1–46) has proven that the symplectic cohomology space of the Galilean group is one-dimensional.

- Lie Algebra of Gallileo Group

$$\begin{bmatrix} \vec{\omega} & \vec{\eta} & \vec{\gamma} \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{bmatrix}, \begin{cases} \vec{\eta} \text{ and } \vec{\gamma} \in R^3, \varepsilon \in R^+ \\ \vec{\omega} \in so(3) : \vec{x} \mapsto \vec{\omega} \times \vec{x} \end{cases}$$

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THALES

Lie Algebra of Lie Group and Adjoint operators

- Let G a Lie Group and T_eG tangent space of G at its neutral element e

  - Ad Adjoint representation of G

$$Ad : G \to GL(T_eG) \quad \text{with} \quad i_g : h \mapsto ghg^{-1}$$

$$g \in G \mapsto Ad_g = T_e i_g$$

  - ad Tangent application of Ad at neutral element e of G

$$ad = T_e Ad : T_eG \to End(T_eG) \qquad X, Y \in T_eG \mapsto ad_X(Y) = [X, Y]$$

- For $G = GL_n(K)$ with $K = R$ or C

$$T_eG = M_n(K) \qquad X \in M_n(K), g \in G \quad Ad_g(X) = gXg^{-1}$$

$$X, Y \in M_n(K) \quad ad_X(Y) = (T_e Ad)_X(Y) = XY - YX = [X, Y]$$

  - Curve from $e = I_d = c(0)$ tangent to $X = c(1)$: $c(t) = \exp(tX)$

    and transform by Ad : $\gamma(t) = Ad \exp(tX)$

$$ad_X(Y) = (T_e Ad)_X(Y) = \frac{d}{dt}\gamma(t)Y\Big|_{t=0} = \frac{d}{dt}\exp(tX)Y\exp(tX)^{-1}\Big|_{t=0} = XY - YX$$

Broken Symmetry and cocycle

- For Classical Thermodynamics, the group G (Group of time translation) leaves unchanged Gibbs equilibrium states.

- This is not true in the general case: the symmetry is broken.

- If we consider a Gibbs state, the probability law μ_β , its image by $\Psi_g(\mu_\beta)$ with $g \in G$ is a probability law.

- To compute $\Psi_g(\mu_\beta)$, we have to compute $J \circ \Psi_g^{-1}$ where J is the moment map $J : x \in M \mapsto U \in \mathfrak{g}^*$

- There exist an application $\theta : G \to \mathfrak{g}^*$ such that:

$$J(\Psi_g(x)) = a(g, J(x)) = Ad_g^*(J(x)) + \theta(g)$$

$$Ad_g^*(U) = U \circ Ad_g^{-1}$$

- θ verifies the equality, proving that it is $\mathfrak{g}^*$-cocycle of Group G

$$\theta(g_1 g_2) = Ad_{g_1}^* \theta(g_2) + \theta(g_1) \ , \ \forall g_1, g_2 \in G$$

$$\theta(e) = 0$$

$$\theta(g^{-1}) = -Ad_g^* \theta(g)$$

Distribution of probability by Group action

The distribution density under the action of the Lie Group is given by:

$$\mu^* : e^{\Phi^* - \langle \beta^*, U \rangle}$$

$$\Phi^* = \Phi(\beta^*) = \Phi - \langle \theta(g^{-1}), \beta \rangle$$

$$\Phi^* = \Phi + \langle \theta(g), Ad_g \beta \rangle$$

(**)

$$\beta^* = Ad_g(\beta)$$

$$\theta(g^{-1}) = -Ad_g^* \theta(g)$$

The set Ω of Geometric Temperature is invariant by the adjoint action of G

$$\Psi_g(\mu_\beta) = \mu_{Ad_g(\beta)}$$

If we use $Q = \frac{\partial \Phi}{\partial \beta}$, we have the constraint $\delta\Phi - \langle Q, \delta\beta \rangle = 0$

By derivation of (**), we have: $\tilde{\Theta}(\beta, Z) + \langle Q, [\beta, Z] \rangle = 0$

$$\tilde{\Theta}(X, Y) : \mathfrak{g} \times \mathfrak{g} \to \Re$$

$$X, Y \mapsto \langle \Theta(X), Y \rangle$$

$$\Theta(X) = T_e \theta(X(e))$$

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Geometric (Planck) Temperature

- We have previously observed that: $\tilde{\Theta}(\beta,Z)+\langle Q,[\beta,Z]\rangle=0$

- $\tilde{\Theta}(X,Y)$ is called the **Symplectic Cocycle of Lie algebra** $\mathfrak{g}$ associated to the momentum map J

$$\tilde{\Theta}(X,Y)=J_{[X,Y]}-\{J_X,J_Y\} \text{ with } \{.,.\} \text{ Poisson Bracket and } J \text{ the Moment Map}$$

  - where J_X linear application from $\mathfrak{g}$ to differential function on M: $\mathfrak{g}\to C^\infty(M,R)$, $X\to J_X$
  - and the associated differentiable application J, called moment(um) map:

$$J:M\to\mathfrak{g}^* \text{ with } x\mapsto J(x) \text{ such that } J_X(x)=\langle J(x),X\rangle,\ X\in\mathfrak{g}$$

- $\tilde{\Theta}(X,Y)$ is a 2-form of $\mathfrak{g}$ and verify:

$$\tilde{\Theta}([X,Y],Z)+\tilde{\Theta}([Y,Z],X)+\tilde{\Theta}([Z,X],Y)=0$$

- If we define: $\tilde{\Theta}_\beta(Z_1,Z_2)=\tilde{\Theta}(Z_1,Z_2)+\langle Q,ad_{Z_1}(Z_2)\rangle$ with $ad_{Z_1}(Z_2)=[Z_1,Z_2]$

- We can observe that : $\beta\in Ker\,\tilde{\Theta}_\beta$ $\tilde{\Theta}_\beta(\beta,\beta)=0\ ,\ \forall\beta\in\mathfrak{g}$

Associated Riemannian Metric: Geometric Fisher Metric

- We can compute the image of Geometric Heat by the Lie Group action:

$$Q^* = Ad_g^*(Q) + \theta(g)$$

- By tangential derivative to the orbite with respect to $Z \in \mathfrak{g}$ and by using positivity of $-\frac{\partial Q}{\partial \beta} \geq 0$, we find:

$$\tilde{\Theta}_\beta(Z,[\beta,Z]) = \tilde{\Theta}(Z,[\beta,Z]) + \langle Q,[Z,[\beta,Z]] \rangle \geq 0$$

- $\tilde{\Theta}_\beta$ is a 2-form of $\mathfrak{g}$ that verifies:

$$\tilde{\Theta}([X,Y],Z) + \tilde{\Theta}([Y,Z],X) + \tilde{\Theta}([Z,X],Y) = 0$$

- Then, there exists a symmetric tensor g_β defined on $ad_\beta(Z)$

$$g_\beta([\beta,Z_1],[\beta,Z_2]) = \tilde{\Theta}_\beta(Z_1,[\beta,Z_2])$$

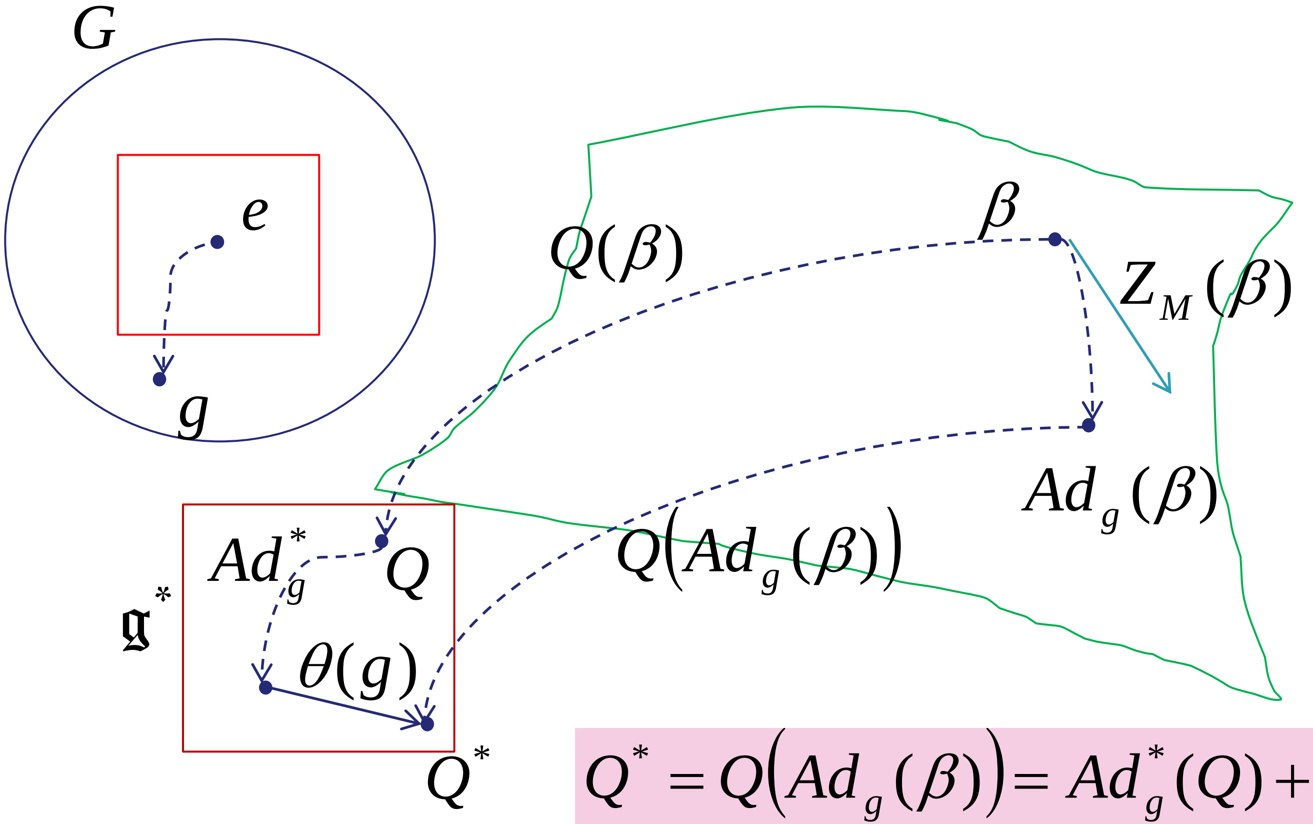
- With the following invariances:

$$I(Ad_g(\beta)) = -\frac{\partial^2\left(\Phi - \langle \theta(g^{-1}),\beta \rangle\right)}{\partial \beta^2} = -\frac{\partial^2 \Phi}{\partial \beta^2} = I(\beta)$$

$$s[Q(Ad_g(\beta))] = s(Q(\beta))$$

THALES

Lie Group Action on Symplectic Manifold



$$Q^* = Q\left(Ad_g(\beta)\right) = Ad_g^*(Q) + \theta(g)$$

Fisher Metric of Souriau Lie Group Thermodynamics

- Souriau has introduced the Riemannian metric

$$g_\beta([\beta,Z_1],[\beta,Z_2])=\tilde{\Theta}_\beta(Z_1,[\beta,Z_2]) \qquad \beta\in Ker\,\tilde{\Theta}_\beta$$

$$\tilde{\Theta}_\beta(Z_1,Z_2)=\tilde{\Theta}(Z_1,Z_2)+\langle Q,ad_{Z_1}(Z_2)\rangle \text{ with } ad_{Z_1}(Z_2)=[Z_1,Z_2]$$

- This metric is an **extension of Fisher metric, an hessian metric**: If we differentiate the relation $Q(Ad_g(\beta))=Ad_g^*(Q)+\theta(g)$

$$\frac{\partial Q}{\partial\beta}(-[Z_1,\beta],.)=\tilde{\Theta}(Z_1,[\beta,.])+\langle Q,Ad_{.Z_1}([\beta,.])\rangle=\tilde{\Theta}_\beta(Z_1,[\beta,.])$$

$$-\frac{\partial Q}{\partial\beta}([Z_1,\beta],Z_2.)=\tilde{\Theta}(Z_1,[\beta,Z_2])+\langle Q,Ad_{.Z_1}([\beta,Z_2])\rangle=\tilde{\Theta}_\beta(Z_1,[\beta,Z_2])$$

$$\Rightarrow -\frac{\partial^2\Phi}{\partial\beta^2}=-\frac{\partial Q}{\partial\beta}=g_\beta([\beta,Z_1],[\beta,Z_2])=\tilde{\Theta}_\beta(Z_1,[\beta,Z_2])$$

- The Fisher Metric is then a **generalization of "Heat Capacity"**:

$$\beta=\frac{1}{kT} \qquad K=-\frac{\partial Q}{\partial\beta}=-\frac{\partial Q}{\partial T}\left(\frac{\partial(1/kT)}{\partial T}\right)^{-1}=kT^2\frac{\partial Q}{\partial T} \qquad \frac{\partial T}{\partial t}=\frac{\kappa}{C.D}\Delta T \text{ with } \frac{\partial Q}{\partial T}=C.D$$

Souriau Moment(um) map

- J_X is a linear application from $\mathfrak{g}$ to differential function on M :

$$\mathfrak{g} \to C^\infty(M,R)$$
$$X \to J_X$$

- the associated differentiable application J , called moment(um) map:

$$J : M \to \mathfrak{g}^* \quad \text{with} \quad x \mapsto J(x) \quad \text{such that } J_X(x) = \langle J(x), X \rangle, X \in \mathfrak{g}$$

- If instead of J , we take the following momentum map:

$$J'(x) = J(x) + Q \ , \ x \in M$$

- where $Q \in \mathfrak{g}^*$ is constant, the symplectic cocycle θ is replaced by

$$\theta'(g) = \theta(g) + Q - Ad_g^* Q$$

- where $\theta' - \theta = Q - Ad_g^* Q$ is one-coboundary of G with values in $\mathfrak{g}^*$.

- We have also properties $\theta(g_1 g_2) = Ad_{g_1}^* \theta(g_2) + \theta(g_1)$ with $\theta(e) = 0$

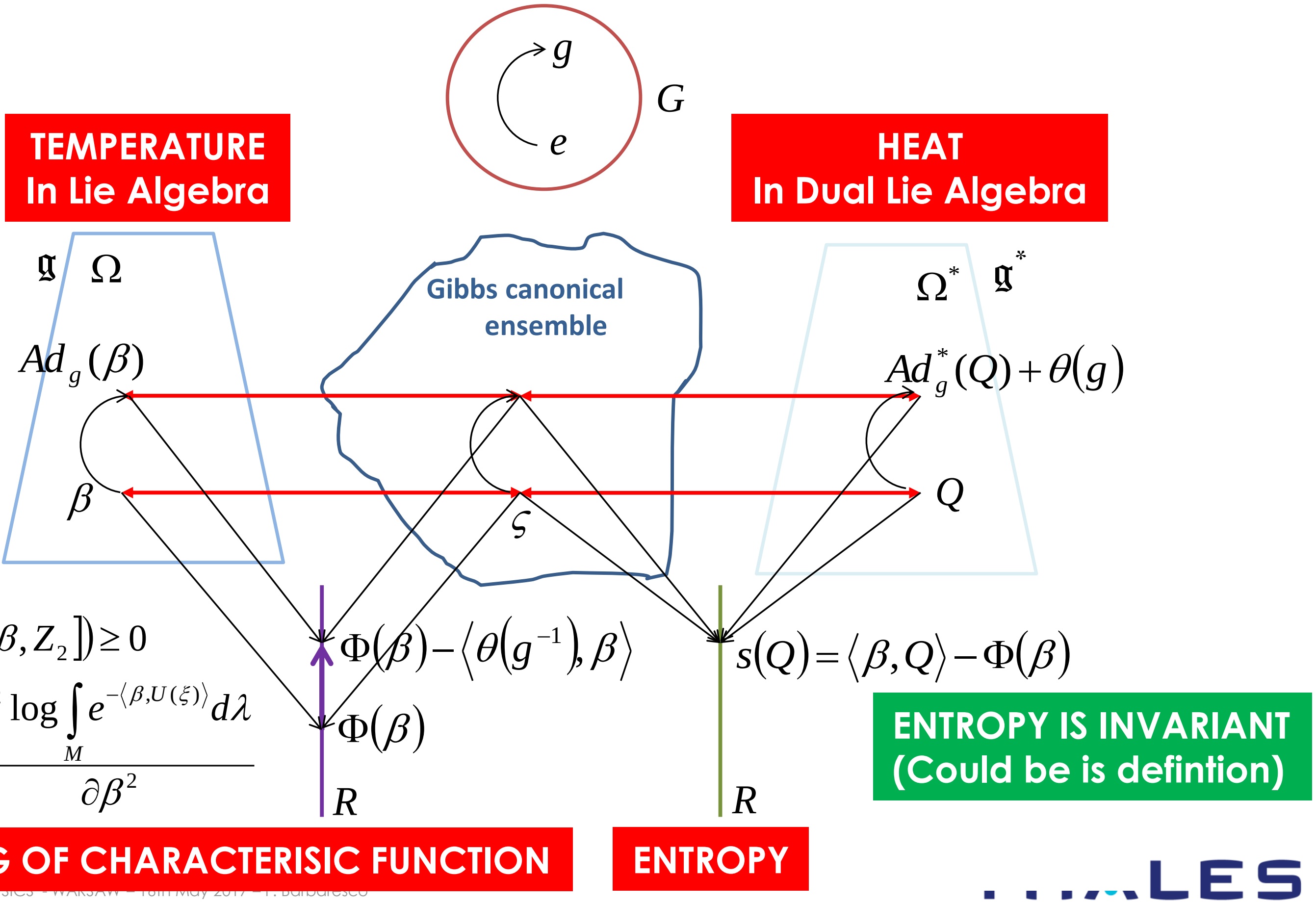
Souriau Theorem of Lie Group Thermodynamics

- Let Ω be the largest open proper subset of $\mathfrak{g}$, Lie algebra of G, such that $\int_M e^{-\langle \beta, U(\xi)\rangle} d\lambda$ and $\int_M \xi . e^{-\langle \beta, U(\xi)\rangle} d\lambda$ are convergent integrals, this set Ω is convex and is invariant under every transformation $Ad_g(.)$, where $g \mapsto Ad_g(.)$ is the adjoint representation of G, such that $Ad_g = T_e i_g$ with $i_g : h \mapsto ghg^{-1}$.

Let $a : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ a unique affine action a such that linear part is coadjoint representation of G, that is the contragradient of the adjoint representation. It associates to each $g \in G$ the linear isomorphism, $Ad_g^* \in GL(\mathfrak{g}^*)$ satisfying, for each $\xi \in \mathfrak{g}^*$, $X \in \mathfrak{g}$: $\left\langle Ad_g^*(\xi), X \right\rangle = \left\langle \xi, Ad_{g^{-1}}(X) \right\rangle$

$\beta \to Ad_g(\beta)$ $\quad$ $\Phi \to \Phi - \theta(g^{-1})\beta$ $\quad$ $s \to s$ $\quad$ $Q \to a(g,Q) = Ad_g^*(Q) + \theta(g)$

Souriau Model of Lie Group Thermodynamics



FISHER METRIC

$$g_\beta([\beta,Z_1],[\beta,Z_2])=\tilde{\Theta}_\beta(Z_1,[\beta,Z_2])\geq 0$$

$$I(\beta)=I(Ad_g(\beta))=-\frac{\partial^2\Phi}{\partial\beta^2}=\frac{\partial^2\log\int_M e^{-\langle\beta,U(\xi)\rangle}d\lambda}{\partial\beta^2}$$

Question

Nous prenons désormais Z dans C . La valeur moyenne du moment $\Psi(x)$ dans l'état de Gibbs est égal à la dérivée

$$Q = z'(Z);$$

$Z \mapsto Q$ est un difféomorphisme analytique de C sur un ouvert convexe de $\mathcal{G}^*$: la transformée de Legendre s de z :

$$s(Q) = QZ - z$$

y est convexe et vérifie $Z = s'(Q)$; la dérivée seconde:

$$K = z''(Z)$$

est un tenseur positif, dont l'inverse est égal à $s''(Q)$.

K munit l'ensemble C d'une structure riemannienne invariante par l'action du groupe; pour cette structure, l'application linéaire $Ad(Z)$ est antihermitienne.

L'application f_Z , définie par:

$$f_Z(Z', Z'') = K([Z, Z'], Z'') \quad \forall Z', Z'' \in \mathcal{G}$$

est un cocycle symplectique, cohomologue à f [formule (2,7 C)]; son noyau est l'orthogonal de l'orbite adjointe de Z pour la structure riemannienne de C .

■ Dans le cas classique, on ne considère que le groupe de dimension 1 des translations temporelles (qui n'est défini qu'après avoir choisi un référentiel - par exemple celui de la boîte qui contient le gaz). Alors, avec des unités convenables, Z est l'inverse de la TEMPERATURE ABSOLUE; z est le POTENTIEL THERMODYNAMIQUE DE PLANCK; $-s$ est l'ENTROPIE; Q est l'ENERGIE INTERNE; K caractérise la CAPACITE CALORIFIQUE. ■

THALES

Question

Ces formules sont universelles, en ce sens qu'elles ne mettent pas en jeu la variété symplectique U - mais seulement le groupe G , son cocycle symplectique f et les couples Θ , Q . Peut-être cette "thermodynamique des groupes de Lie" a-t-elle un intérêt mathématique.

THALES

Souriau Lie Group Thermodynamics

Si nous désignons par E l'énergie généralisée (c'est-à-dire le moment qui appartient, rappelons-le, au dual de l'algèbre de Lie $\mathfrak{g}$ du groupe), nous écrirons

(7.1) $$\rho = e^{E.\Theta - z}$$

Θ étant un élément de $\mathfrak{g}$ qui indexe l'état statistique (et qui va généraliser la température), z une constante de normalisation, que l'on détermine en écrivant que la masse de la loi de probabilité est égale à 1 :

(7.2) $$z = \log \int_U e^{E\Theta} d\lambda(x)$$

Souriau Lie Group Thermodynamics: Geometric Calorific Capacity

Il faut bien entendu que cette intégrale soit convergente ; nous définirons l'ensemble canonique de Gibbs Ω comme le plus grand ouvert (dans l'algèbre de Lie) où cette intégrale est localement normalement convergente (en Θ). On montre que Ω est convexe, et que z est une fonction C^∞ sur Ω ; que la dérivée $Q = \frac{\partial z}{\partial \Theta}$ coïncide avec la valeur moyenne de l'énergie E (Q généralise donc la chaleur) ; que le tenseur $\frac{\partial Q}{\partial \Theta}$ est symétrique et positif (il généralise la capacité calorifique). Il en résulte que z est fonction convexe de Θ ; la transformation de Legendre lui associe une fonction concave, à savoir

$$Q \mapsto s = z - Q\Theta \qquad (7.3)$$

s est l'entropie.

Fisher-Souriau Metric based on cocycle

pour chaque "température" Θ , définissons un tenseur f_Θ , somme du cocycle f (défini en (3.2)) et du cobord de la chaleur :

(7.4) $$f_\Theta(Z,Z') = f(Z,Z') + Q[Z,Z'] .$$

f_Θ jouit alors des propriétés suivantes :

5)
a) f_Θ est un cocycle symplectique ;

b) $\Theta \in \ker f_\Theta$

c) Le tenseur symétrique g_Θ , défini sur l'ensemble de valeurs de $ad(\Theta)$ par

$$g_\Theta([\Theta,Z],[\Theta,Z']) = f_\Theta(Z,[\Theta,Z']) .$$

est positif (et même défini positif si l'action du groupe est effective).

Ces formules sont universelles, en ce sens qu'elles ne mettent pas en jeu la variété symplectique U - mais seulement le groupe G , son cocycle symplectique f et les couples Θ , Q . Peut-être cette "thermodynamique des groupes de Lie" a-t-elle un intérêt mathématique.

Lie Group Thermodynamics: Centrifuge for Butter, U235 & Ribo acides

- Physiquement, la théorie donne de bons résultats si on l'applique aux divers sous-groupes du groupe de Galilée qui sont caractéristiques des appareils thermodynamiques : ainsi une boite cylindrique dans laquelle on enferme un fluide lui laisse un sous-groupe d'invariance de dimension 2 : rotations autour de l'axe, translations temporelles. D'où résulte un vecteur température à deux dimensions, que l'on peut "transmettre" au fluide par l'intermédiaire de la boite, (en la refroidissant, par exemple, et en la faisant tourner) ; les résultats de la théorie sont ceux-là même que l'on exploite dans les centrifugeuses (par exemple pour fabriquer du beurre, de l'uranium 235 ou des acides ribonucléiques).

- On remarquera que le processus par lequel une centrifugeuse réfrigérée transmet son propre vecteur-température à son contenu porte deux noms différents : conduction thermique et viscosité, selon la composante du vecteur-température que l'on considère ; conduction et viscosité devraient donc être unifiées dans une théorie fondamentale des processus irréversibles (théorie qui reste à construire).

Mean Value could be only defined in case of affine structure !!

Souriau Citation:

- We can only measure mean values on objects belonging to a set with physical affine structure and this structure should be unique !!!!
- « Il est évident que l'on ne peut définir de valeurs moyennes que sur des objets appartenant à un espace vectoriel (ou affine); donc—si bourbakiste que puisse sembler cette affirmation—que l'on n'observera et ne mesurera de valeurs moyennes que sur des grandeurs appartenant à un ensemble possédant physiquement une structure affine. Il est clair que cette structure est nécessairement unique—sinon les valeurs moyennes ne seraient pas bien définies. »

Continuous Medium Thermodynamics

- For Continuous Medium Thermodynamics, « Temperature Vector » is no longer constrained to be in Lie Algebra, but only contrained by phenomenologic equations (e.g. Navier equations, …).

- For Thermodynamic equilibrium, the **« Temperature Vector » is a Killing vector of Space-Time.**

- For each point X, there is a « Temperature Vector » $\beta(X)$, such it is an infinitesimal conformal transform of the metric of the univers g_{ij} :

$$\hat{\partial}_i \beta_j + \hat{\partial}_j \beta_i = \lambda g_{ij}$$
$$\partial_i \beta_j + \partial_j \beta_i - 2\Gamma_{ij}^k \beta_k = \lambda g_{ij}$$
$$\lambda = 0 \Rightarrow \text{Killing Equation}$$

$$\begin{cases} \hat{\partial}_i . : \text{covariant derivative} \\ \beta_j : \text{component of Temperature vector} \end{cases}$$

- Conservation equations can be deduced for components of Impulsion-Energy tensor T^{ij} and Entropy flux S^j : $\hat{\partial}_i T^{ij} = 0 \quad \partial_i S^j = 0$

Comparison of Affine Representation of Lie Group and Lie Algebra in Souriau an Koszul works

OPEN

Affine representation of Lie group and Lie algebra by Souriau

- Souriau called the Mechanics deduced from his model: "**Affine Mechanics**"

- Let *G* be a Lie group and *E* a finite-dimensional vector space. A map

$A : G \to Aff(E)$ can always be written as:

$$A(g)(x) = R(g)(x) + \theta(g) \text{ with } g \in G, x \in E$$

where the maps $R : G \to GL(E)$ and $\theta : G \to E$ are determined by *A*. The map *A* is an affine representation of *G* in *E*.

- The map $\theta : G \to E$ is a one-cocycle of *G* with values in *E*, for the linear representation *R*; it means that θ is a smooth map which satisfies, for all $g, h \in G$:

$$\theta(gh) = R(g)(\theta(h)) + \theta(g)$$

Affine representation of Lie group and Lie algebra by Souriau

- Let $\mathfrak{g}$ be a Lie algebra and *E* a finite-dimensional vector space. A linear map $a:\mathfrak{g}\rightarrow aff(E)$ always can be written as:

$$a(X)(x)=r(X)(x)+\Theta(X) \quad \text{with} \quad X\in\mathfrak{g}, x\in E$$

- where the linear maps $r:\mathfrak{g}\rightarrow gl(E)$ and $\Theta:\mathfrak{g}\rightarrow E$ are determined by *a*. The map *a* is an affine representation of *G* in *E*.

- The linear map $\Theta:\mathfrak{g}\rightarrow E$ is a one-cocycle of *G* with values in *E*, for the linear representation *r*; it means that Θ satisfies, for all $X,Y\in\mathfrak{g}$:

$$\Theta([X,Y])=r(X)(\Theta(Y))-r(Y)(\Theta(X))$$

- Θ is called the one-cocycle of $\mathfrak{g}$ associated to the affine representation *a*.

- the associated cocycle $\Theta:\mathfrak{g}\rightarrow E$ is related to the one-cocycle $\theta:G\rightarrow E$ by:

$$\Theta(X)=T_e\theta(X(e)) \ , \quad X\in\mathfrak{g}$$

Equivariance of Souriau Moment Map

- There exists a unique affine action a such that the linear part is a coadjoint representation:

$$a : G \times \mathfrak{g}^* \to \mathfrak{g}^*$$

$$a(g,\xi) = Ad^*_{g^{-1}}\xi + \theta(g)$$

with $\left\langle Ad^*_{g^{-1}}\xi, X \right\rangle = \left\langle \xi, Ad_{g^{-1}}X \right\rangle$

- that induce equivariance of moment J .

Action of Lie Group on a Symplectic Manifold

- Let $\Phi : G \times M \to M$ be an action of Lie Group G on differentiable manifold M, the fundamental field associated to an element X of Lie algebra $\mathfrak{g}$ of group G is the vectors field X_M on M:

$$X_M(x) = \frac{d}{dt}\Phi_{\exp(-tX)}(x)\Big|_{t=0}$$ with $\Phi_{g_1}\left(\Phi_{g_2}(x)\right) = \Phi_{g_1 g_2}(x)$ and $\Phi_e(x) = x$

- Φ is Hamiltonian on a symplectic manifold M, if Φ is symplectic and if for all $X \in \mathfrak{g}$, the fundamental field X_M is globally Hamiltonian.

- There is a unique action *a* of the Lie group G on the dual $\mathfrak{g}^*$ of its Lie algebra for which the moment map J is equivariant, that means satisfies for each $x \in M$:

$$J\left(\Phi_g(x)\right) = a(g, J(x)) = Ad^*_{g^{-1}}\left(J(x)\right) + \theta(g)$$

$$\langle T_e\theta(X), Y\rangle = \langle \Theta(X), Y\rangle = \tilde{\Theta}(X, Y) = J_{[X,Y]} - \{J_X, J_Y\}$$

$$\tilde{\Theta} : \mathfrak{g} \times \mathfrak{g} \to \Re$$

$$\tilde{\Theta}([X,Y],Z) + \tilde{\Theta}([Y,Z],X) + \tilde{\Theta}([Z,X],Y) = 0$$

THALES

Affine representation of Lie group and Lie algebra by Koszul

- Let Ω be a convex domain in R^n containing no complete straight lines, and an associated convex cone $V(\Omega)=\{(\lambda x, x)\in R^n\times R / x\in\Omega, \lambda\in R^+\}$
Then there exists an affine embedding:

$$\ell : x\in\Omega \mapsto \begin{bmatrix} x \\ 1 \end{bmatrix} \in V(\Omega)$$

- If we consider η the group of homomorphism of $A(n,R)$ into $GL(n+1,R)$ given by:

$$s\in A(n,R)\mapsto \begin{bmatrix} \mathbf{f}(s) & \mathbf{q}(s) \\ 0 & 1 \end{bmatrix} \in GL(n+1,R)$$

- and associated affine representation of Lie Algebra:

$$\begin{bmatrix} f & q \\ 0 & 0 \end{bmatrix}$$

- with $A(n,R)$ the group of all affine transformations of R^n. We have

$\eta(G(\Omega))\subset G(V(\Omega))$ and the pair (η,ℓ) of the homomorphism

$\eta : G(\Omega)\to G(V(\Omega))$ and the map $\ell:\Omega\to V(\Omega)$ is equivariant.

Affine representation of Lie group and Lie algebra by Koszul

- Let G a connex Lie Group and E a real or complex vector space of finite dimension, Koszul has introduced an affine representation of G in E such that the following is an affine transformation: $E \rightarrow E$

$$a \mapsto sa \quad \forall s \in G$$

- We set $A(E)$ the set of all affine transformations of a vector space E, a Lie Group called affine transformation group of E. The set $GL(E)$ of all regular linear transformations of E, a subgroup of $A(E)$.

- We define a linear representation from E to $GL(E)$:

$$\mathbf{f} : G \rightarrow GL(E)$$
$$s \mapsto \mathbf{f}(s)a = sa - so \quad \forall a \in E$$

- and an application from G to E:

$$\mathbf{q} : G \rightarrow E$$
$$s \mapsto \mathbf{q}(s) = so \quad \forall s \in G$$

- Then we have $\forall s,t \in G$: $\mathbf{f}(s)\mathbf{q}(t) + \mathbf{q}(s) = \mathbf{q}(st)$

$$\mathbf{f}(s)\mathbf{q}(t) + \mathbf{q}(s) = s\mathbf{q}(t) - so + so = s\mathbf{q}(t) = sto = \mathbf{q}(st)$$

Affine representation of Lie group and Lie algebra by Koszul

- On the contrary, if an application $\mathbf{q}$ from G to E and a linear representation $\mathbf{f}$ from G to $GL(E)$ verify previous equation, then we can define an affine representation of G in E, written $(\mathbf{f},\mathbf{q})$:

$$Aff(s): a \mapsto sa = \mathbf{f}(s)a + \mathbf{q}(s) \quad \forall s \in G, \forall a \in E$$

- The condition $\mathbf{f}(s)\mathbf{q}(t) + \mathbf{q}(s) = \mathbf{q}(st)$ is equivalent to requiring the following mapping to be an homomorphism: $Aff : s \in G \mapsto Aff(s) \in A(E)$

- We write f the linear representation of Lie algebra $\mathfrak{g}$ of G, defined by $\mathbf{f}$ and q the restriction to $\mathfrak{g}$ of the differential to $\mathbf{q}$ (f and q the differential of $\mathbf{f}$ and $\mathbf{q}$ respectively), Koszul has proved that:

$$f(X)q(Y) - f(Y)q(X) = q([X,Y]) \quad \forall X,Y \in \mathfrak{g}$$

$$\text{with} \quad f:\mathfrak{g} \rightarrow gl(E) \quad \text{and} \quad q:\mathfrak{g} \mapsto E$$

Where $gl(E)$ the set of all linear endomorphisms of E, the Lie algebra of $GL(E)$

Affine representation of Lie group and Lie algebra by Koszul

- Conversely, if we assume that $\mathfrak{g}$ admits an affine representation (f,q) on E, using an affine coordinate system $\{x^1,...,x^n\}$ on E, we can express an affine mapping $v \mapsto f(X)v + q(Y)$ by an $(n+1)\times(n+1)$ matrix representation:

$$aff(X) = \begin{bmatrix} f(X) & q(X) \\ 0 & 0 \end{bmatrix}$$

- where $f(X)$ is a $n \times n$ matrix and $q(X)$ is a n row vector.

- If we denote $\mathfrak{g}_{aff} = aff(\mathfrak{g})$, we write G_{aff} the linear Lie subgroup of $GL(n+1,R)$ generated by $\mathfrak{g}_{aff}$. An element of $s \in G_{aff}$ is expressed by:

$$Aff(s) = \begin{bmatrix} \mathbf{f}(s) & \mathbf{q}(s) \\ 0 & 1 \end{bmatrix}$$

Souriau/Koszul: Affine Representation of Lie Group and Lie Algebra

Souriau Model of Affine Representation of Lie Groups and Algebra	Koszul Model of Affine Representation of Lie Groups and Algebra
$A(g)(x) = R(g)(x) + \theta(g)$ with $g \in G, x \in E$ $R : G \to GL(E)$ and $\theta : G \to E$	$Aff(s) : a \mapsto sa = \mathbf{f}(s)a + \mathbf{q}(s) \quad \forall s \in G, \forall a \in E$ $\mathbf{f} : G \to GL(E)$ $s \mapsto \mathbf{f}(s)a = sa - so \quad \forall a \in E$ $\mathbf{q} : G \to E$ $s \mapsto \mathbf{q}(s) = so \quad \forall s \in G$
$\theta(gh) = R(g)(\theta(h)) + \theta(g)$ with $g, h \in G$ $\theta : G \to E$ is a one-cocycle of G with values in E,	$\mathbf{q}(st) = \mathbf{f}(s)\mathbf{q}(t) + \mathbf{q}(s)$
$a(X)(x) = r(X)(x) + \Theta(X)$ with $X \in \mathfrak{g}, x \in E$ The linear map $\Theta : \mathfrak{g} \to E$ is a one-cocycle of G with values in E: $\Theta(X) = T_e\theta(X(e))$, $X \in \mathfrak{g}$	$v \mapsto f(X)v + q(Y)$ f and q the differential of $\mathbf{f}$ and $\mathbf{q}$ respectively
$\Theta([X,Y]) = r(X)(\Theta(Y)) - r(Y)(\Theta(X))$	$q([X,Y]) = f(X)q(Y) - f(Y)q(X) \quad \forall X, Y \in \mathfrak{g}$ with $f : \mathfrak{g} \to gl(E)$ and $q : \mathfrak{g} \mapsto E$
none	$aff(X) = \begin{bmatrix} f(X) & q(X) \\ 0 & 0 \end{bmatrix}$
none	$Aff(s) = \begin{bmatrix} \mathbf{f}(s) & \mathbf{q}(s) \\ 0 & 1 \end{bmatrix}$

THALES

THALES

Euler-Lagrange Equation of Lie Group Thermodynamics

OPEN

Seminal Paper of Poincaré 1901 on « Euler-Poincaré Equation »

[1] Henri Poincaré, Sur une forme nouvelle des équations de la Mécanique, C. R. Acad. Sci. Paris, T. CXXXII, n. 7, p. 369–371., 1901

- Heni Poincaré proved that when a Lie algebra acts locally transitively on the configuration space of a Lagrangian mechanical system, the Euler-Lagrange equations are equivalent to a new system of differential equations defined on the product of the configuration space with the Lie algebra

SÉANCE DU LUNDI 18 FÉVRIER 1901,

PRÉSIDENCE DE M. FOUQUÉ.

MEMOIRES ET COMMUNICATIONS

DES MEMBRES ET DES CORRESPONDANTS DE L'ACADÉMIE.

MÉCANIQUE RATIONNELLE. — *Sur une forme nouvelle des équations de la Mécanique*. Note de M. H. Poincaré.

« Ayant eu l'occasion de m'occuper du mouvement de rotation d'un corps solide creux, dont la cavité est remplie de liquide, j'ai été conduit à mettre les équations générales de la Mécanique sous une forme que je crois nouvelle et qu'il peut être intéressant de faire connaître.

$$\frac{d}{dt}\frac{dT}{d\eta_s} = \sum c_{ski}\frac{dT}{d\eta_i}\eta_k + \Omega_s.$$

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THALES

Euler-Poincaré Equation of Lie Group Thermodynamics

- When a Lie algebra acts locally transitively on the configuration space of a Lagrangian mechanical system, Henri Poincaré proved that the Euler-Lagrange equations are equivalent to a new system of differential equations defined on the product of the configuration space with the Lie algebra.

- Euler-Poincaré equations can be written under an intrinsic form, without any reference to a particular system of local coordinates, proving that they can be conveniently expressed in terms of the Legendre and momentum maps of the lift to the cotangent bundle of the Lie algebra action on the configuration space.

Euler-Poincaré Equation of Lie Group Thermodynamics

- The Lagrangian is a smooth real valued function L defined on the tangent bundle TM. To each parameterized continuous, piecewise smooth curve , $\gamma:[t_0,t_1]\to M$,defined on a closed interval $[t_0,t_1]$, with values in M , one associates the value at γ of the action integral: $I(\gamma)=\int_{t_0}^{t_1} L\left(\frac{d\gamma(t)}{dt}\right)dt$
- The partial differential of the function $L:M\times\mathfrak{g}\to\Re$ with respect to its second variable $d_2\overline{L}$, which plays an important part in the Euler-Poincaré equation, can be expressed in terms of the momentum and Legendre maps: $d_2\overline{L}=p_{\mathfrak{g}^*}\circ\varphi^t\circ\mathbb{L}\circ\varphi$ with $J=p_{\mathfrak{g}^*}\circ\varphi^t(\Rightarrow d_2\overline{L}=J\circ\mathbb{L}\circ\varphi)$ the moment map, $p_{\mathfrak{g}^*}:M\times\mathfrak{g}^*\to\mathfrak{g}^*$
- the canonical projection on the 2nd factor, $\mathbb{L}:TM\to T^*M$ the Legendre transform, with $\varphi:M\times\mathfrak{g}\to TM\ /\ \varphi(x,X)=X_M(x)$

and $\varphi^t:T^*M\to M\times\mathfrak{g}^*\ /\ \varphi^t(\xi)=\left(\pi_M(\xi),J(\xi)\right)$

Euler-Poincaré Equation of Lie Group Thermodynamics

- The Euler-Poincaré equation can therefore be written under the form:

$$\left(\frac{d}{dt} - ad^*_{V(t)}\right)\left(d_2\bar{L}(\gamma(t),V(t))\right) = J \circ d_1\bar{L}(\gamma(t),V(t))$$

$$\left(\frac{d}{dt} - ad^*_{V(t)}\right)\left(J \circ \mathbb{L} \circ \varphi(\gamma(t),V(t))\right) = J \circ d_1\bar{L}(\gamma(t),V(t)) \text{ with } \frac{d\gamma(t)}{dt} = \varphi(\gamma(t),V(t))$$

$$H(\xi) = \left\langle \xi, \mathbb{L}^{-1}(\xi) \right\rangle - L\left(\mathbb{L}^{-1}(\xi)\right) \,,\; \xi \in T^*M \;,\; \mathbb{L} : TM \to T^*M \;,\; H : T^*M \to R$$

- Following the remark made by Poincaré at the end of his note, the most interesting case is when the map $\bar{L} : M \times \mathfrak{g} \to R$ only depends on its second variable $X \in \mathfrak{g}$. The Euler-Poincaré equation becomes:

$$\left(\frac{d}{dt} - ad^*_{V(t)}\right)\left(d\bar{L}(V(t))\right) = 0 \qquad d\bar{L}(V(t)) = d_2\bar{L}(\gamma(t),V(t))$$

Euler-Poincaré Equation of Lie Group Thermodynamics

- We can use analogy of structure when the convex Gibbs ensemble is homogeneous. We can then apply Euler-Poincaré equation for Lie Group Thermodynamics. Considering Clairaut equation:

$$s(Q)=\langle\beta,Q\rangle-\Phi(\beta)=\left\langle\Theta^{-1}(Q),Q\right\rangle-\Phi\left(\Theta^{-1}(Q)\right)$$ with $Q=\Theta(\beta)=\frac{\partial\Phi}{\partial\beta}\in\mathfrak{g}^*$, $\beta=\Theta^{-1}(Q)\in\mathfrak{g}$

- A Souriau-Euler-Poincaré equation can be elaborated for Souriau Lie Group Thermodynamics:

$$\frac{dQ}{dt}=ad_\beta^*Q$$ and $$\frac{d}{dt}\left(Ad_g^*Q\right)=0$$

New interesting Equations for Thermodynamics

- An associated equation on Entropy is:

$$\frac{ds}{dt}=\left\langle\frac{d\beta}{dt},Q\right\rangle+\left\langle\beta,ad_\beta^*Q\right\rangle-\frac{d\Phi}{dt}$$

- That reduces to: $$\frac{ds}{dt}=\left\langle\frac{d\beta}{dt},Q\right\rangle-\frac{d\Phi}{dt}$$

Due to

$$\langle\xi,ad_V X\rangle=-\langle ad_V^*\xi,X\rangle\Rightarrow\left\langle\beta,ad_\beta^*Q\right\rangle=\left\langle Q,ad_\beta\beta\right\rangle=0$$

Poincaré-Cartan Integral Invariant of Lie Group Thermodynamics

- Analogies between Geometric Mechanics & Geometric Lie Group Thermodynamics, provides the following similarities of structures:

$$\begin{cases} \dot{q} \leftrightarrow \beta \\ p \leftrightarrow Q \end{cases} \qquad \begin{cases} L(\dot{q}) \leftrightarrow \Phi(\beta) \\ H(p) \leftrightarrow s(Q) \\ H = p.\dot{q} - L \leftrightarrow s = \langle Q, \beta \rangle - \Phi \end{cases} \qquad \begin{cases} \dot{q} = \dfrac{dq}{dt} = \dfrac{\partial H}{\partial p} \leftrightarrow \beta = \dfrac{\partial s}{\partial Q} \\ p = \dfrac{\partial L}{\partial \dot{q}} \leftrightarrow Q = \dfrac{\partial \Phi}{\partial \beta} \end{cases}$$

- We can then consider a similar Poincaré-Cartan-Souriau Pfaffian form:

$$\omega = p.dq - H.dt \leftrightarrow \omega = \langle Q, (\beta.dt) \rangle - s.dt = \left(\langle Q, \beta \rangle - s \right).dt = \Phi(\beta).dt$$

- This analogy provides an associated Poincaré-Cartan Integral Invariant:

$$\int_{C_a} p.dq - H.dt = \int_{C_b} p.dq - H.dt \quad \text{transforms in} \quad \int_{C_a} \Phi(\beta).dt = \int_{C_b} \Phi(\beta).dt$$

- For Thermodynamics, we can then deduce an Euler-Poincaré Variational Principle: The Variational Principle holds on $\mathfrak{g}$, for variations $\delta\beta = \dot{\eta} + [\beta, \eta]$, where $\eta(t)$ is an arbitrary path that vanishes at the endpoints, $\eta(a) = \eta(b) = 0$:

$$\delta \int_{t_0}^{t_1} \Phi(\beta(t)).dt = 0$$

THALES

Compatibility with Gauge Model of Balian-Valentin

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Compatible Balian Gauge Theory of Thermodynamics

- Entropy S is an extensive variable $q^0 = S(q^1,...,q^n)$ depending on $q^i \ (i=1,...,n)$ n independent extensive/conservative quantities characterizing the system

- The n intensive variables γ_i are defined as the partial derivatives:

$$\gamma_i = \frac{\partial S(q^1,...,q^n)}{\partial q^i}$$

- Balian has introduced a non-vanishing gauge variable which multiplies all the intensive variables, defining a new set of variables: $p_i = -p_0.\gamma_i \quad , \quad i=1,...,n$

- The 2n+1-dimensional space is thereby extended into a 2n+2-dimensional thermodynamic space T spanned by the variables p_i, q^i with $i = 0,1,...,n$,where the physical system is associated with a n+1-dimensional manifold M in T , parameterized for instance by the coordinates $q^1,...,q^n$ and p_0 .

Compatible Balian Gauge Theory of Thermodynamics

- the contact structure in 2n+1 dimension: $\tilde{\omega} = dq^0 - \sum_{i=1}^{n} \gamma_i . dq^i$
- is embedded into a symplectic structure in 2n+2 dimension, with 1-form, as symplectization: $\omega = \sum_{i=0}^{n} p_i . dq^i$
- The n +1-dimensional thermodynamic manifolds M are characterized by : $\omega = 0$. The 1-form induces then a symplectic structure on T : $d\omega = \sum_{i=0}^{n} dp_i \wedge dq^i$
- The concavity of the entropy $S(q^1,...,q^n)$ as function of the extensive variables, expresses the stability of equilibrium states. It entails the existence of a metric structure in the n-dimensional space q_i :

$$ds^2 = -d^2 S = -\sum_{i,j=1}^{n} \frac{\partial^2 S}{\partial q^i \partial q^j} dq^i dq^j$$

- which defines a distance between two neighboring thermodynamic states:

$$d\gamma_i = \sum_{j=1}^{n} \frac{\partial^2 S}{\partial q^i \partial q^j} dq^j \qquad ds^2 = -\sum_{i=1}^{n} d\gamma_i dq_i = \frac{1}{p_0} \sum_{i=0}^{n} dp_i dq^i$$

Compatible Balian Gauge Theory of Thermodynamics

- **We can observe that this Gauge Theory of Thermodynamics is compatible with Souriau Lie Group Thermodynamics, where we have to consider the Souriau vector :**

$$\beta = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{bmatrix}$$ **transformed in a new vector** $$p_i = -p_0 . \gamma_i$$

$$p = \begin{bmatrix} -p_0\gamma_1 \\ \vdots \\ -p_0\gamma_n \end{bmatrix} = -p_0 . \beta$$

Links with Natural Exponential Families Invariant by a Group: Casilis and Letac

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NEF (Natural Exponential Families): Letac & Casalis

- Let E a vector space of finite size, E^* its dual. $\langle \theta, x \rangle$ Duality braket with $(\theta, x) \in E^* \times E$. μ Positive Radon measure on E, Laplace transform is :

$$L_\mu : E^* \to [0, \infty] \text{ with } \theta \mapsto L_\mu(\theta) = \int_E e^{\langle \theta, x \rangle} \mu(dx)$$

- Transformation $k_\mu(\theta)$ defined on $\Theta(u)$ interior of $D_\mu = \{\theta \in E^*, L_\mu < \infty\}$

$$k_\mu(\theta) = \log L_\mu(\theta)$$

- Natural exponential families are given by:

$$F(\mu) = \left\{ P(\theta, \mu)(dx) = e^{\langle \theta, x \rangle - k_\mu(\theta)} \mu(dx), \theta \in \Theta(\mu) \right\}$$

- Injective function (domian of means): $k'_\mu(\theta) = \int_E x P(\theta, \mu) \mu(dx)$
- And the inverse function: $\psi_\mu : M_F \to \Theta(\mu)$ with $M_F = \mathrm{Im}\left(k'_\mu(\Theta(\mu))\right)$
- Covariance operator: $V_F(m) = k''_\mu(\psi_\mu(m)) = \left(\psi'_\mu(m)\right)^{-1}, \ m \in M_F$

NEF (Natural Exponential Families): Letac & Casalis

- Measure generetad by a familly F :

$$F(\mu)=F(\mu')\Leftrightarrow \exists(a,b)\in E^{*}\times R,\text{such that }\ \mu'(dx)=e^{\langle a,x\rangle+b}\mu(dx)$$

- Let F an exponential family of E generated by μ and $\varphi : x \mapsto g_{\varphi}x+v_{\varphi}$

with $g_{\varphi}\in GL(E)$ automorphisms of E and $v_{\varphi}\in E$, then the familly

$\varphi(F)=\{\varphi(P(\theta,\mu)),\theta\in\Theta(\mu)\}$ is an exponential familly of E

generated by $\varphi(\mu)$

- <u>Definition:</u> An exponential familly F is invariant by a group G (affine group of E), if $\forall\varphi\in G,\varphi(F)=F$: $\forall\mu, F(\varphi(\mu))=F(\mu)$

(the contrary could be false)

NEF (Natural Exponential Families): Letac & Casalis

- Theorem (Casalis): Let $F = F(\mu)$ an exponential familly of E and G affine group of E, then F is invariant by G if and only:

$$\exists a : G \to E^*, \exists b : G \to R, \text{such that :}$$

$$\forall (\varphi, \varphi') \in G^2, \begin{cases} a(\varphi\varphi') = {}^t g_\varphi^{-1} a(\varphi') + a(\varphi) \\ b(\varphi\varphi') = b(\varphi) + b(\varphi') - \langle a(\varphi'), g_\varphi^{-1} v_\varphi \rangle \end{cases}$$

$$\forall \varphi \in G, \varphi(\mu)(dx) = e^{\langle a(\varphi), x \rangle + b(\varphi)} \mu(dx)$$

- When G is a linear subgroup, b is a character of G, a could be obtained by the help of Cohomology of Lie groups .

NEF (Natural Exponential Families): Letac & Casalis

- If we define action of G on E^* by: $g.x = {}^t g^{-1} x, g \in G, x \in E^*$

  we can verify that: $a(g_1 g_2) = g_1.a(g_2) + a(g_1)$

- the action a is an inhomogeneous 1-cocycle: $\forall n > 0$, let the set of all functions from G^n to E^*, $\Im(G^n, E^*)$ called inhomogenesous n-cochains, then we can define the operators: $d^n : \Im(G^n, E^*) \to \Im(G^{n+1}, E^*)$

$$d^n F(g_1, \cdots, g_{n+1}) = g_1.F(g_2, \cdots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i F(g_1, g_2, \cdots, g_i g_{i+1}, \cdots, g_n)$$
$$+ (-1)^{n+1} F(g_1, g_2, \cdots, g_n)$$

NEF (Natural Exponential Families): Letac & Casalis

- Let $Z^n(G,E^*)=Ker(d^n), B(G,E^*)=\mathrm{Im}(d^{n-1})$, with Z^n inhomogneous n-cocycles , the quotient $H^n(G,E^*)=Z^n(G,E^*)/B^n(G,E^*)$ is the **Cohomology Group** of G with value in E^*. We have:

$$d^0:E^*\to\Im(G,E^*) \qquad Z^0=\{x\in E^*;g.x=x,\forall g\in G\}$$

$$x\mapsto(g\mapsto g.x-x)$$

$$d^1:\Im(G,E^*)\to\Im(G^2,E^*)$$

$$F\mapsto d^1F\ ,\ d^1F(g_1,g_2)=g_1.F(g_2)-F(g_1g_2)+F(g_1)$$

$$Z^1=\{F\in\Im(G,E^*);F(g_1g_2)=g_1.F(g_2)+F(g_1),\forall(g_1,g_2)\in G^2\}$$

$$B^1=\{F\in\Im(G,E^*);\exists x\in E^*,F(g)=g.x-x\}$$

NEF (Natural Exponential Families): Letac & Casalis

- When the Cohomology Group $H^1(G,E^*)=0$ then $Z^1(G,E^*)=B^1(G,E^*)$

$$\Rightarrow \exists c \in E^*, \text{such that } \forall g \in G, a(g)=\left(I_d - {}^t g^{-1}\right)c$$

Then if $F=F(\mu)$ is an exponential familly invariant by G , μ verifies

$$\forall g \in G, g(\mu)(dx)=e^{\langle c,x\rangle - \langle c, g^{-1}x\rangle + b(g)}\mu(dx)$$

$$\forall g \in G, g\left(e^{\langle c,x\rangle}\mu(dx)\right)=e^{b(g)}e^{\langle c,x\rangle}\mu(dx) \text{ with } \mu_0(dx)=e^{\langle c,x\rangle}\mu(dx)$$

- For all compact Group, $H^1(G,E^*)=0$ and we can express a

$$A: G \rightarrow GA(E) \qquad \forall (g,g') \in G^2, A_{gg'}=A_g A_{g'}$$

$$g \mapsto A_g \; , \; A_g(\theta)={}^t g^{-1}\theta + a(g) \qquad A(G) \text{ compact sub-group of } GA(E)$$

$$\exists \text{fixed point} \Rightarrow \forall g \in G, A_g(c)={}^t g^{-1}c + a(g)=c \Rightarrow a(g)=\left(I_d - {}^t g^{-1}\right)c$$

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LIE GROUP For Exponential Families in Information Geometry

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www.thalesgroup.com

Lie Group, Lie Algebra, Dual Lie Algebra

Arnold, Vladimir, Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. Annales de l'institut Fourier, 16 no. 1 (1966), p. 319-361:
http://archive.numdam.org/article/AIF_1966__16_1_319_0.pdf

Vladimir Arnold Joke:
« Dans ce qui suit, j'ai tâché conformément à l'appel de N. Bourbaki, de substituer toujours les calculs aveugles aux idées lucides d'Euler »

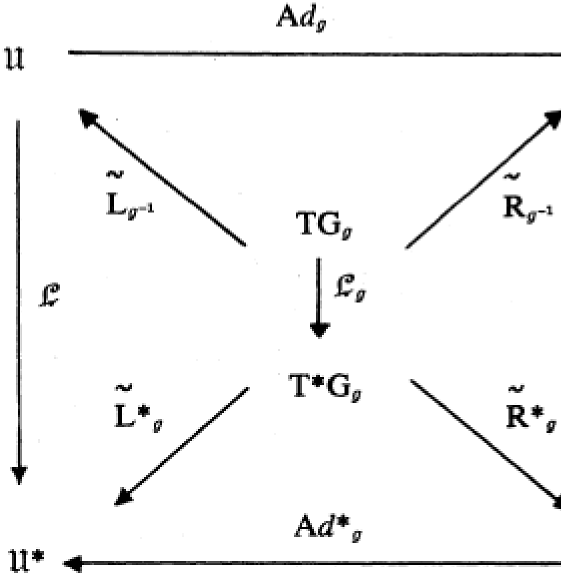

Diagramme D.

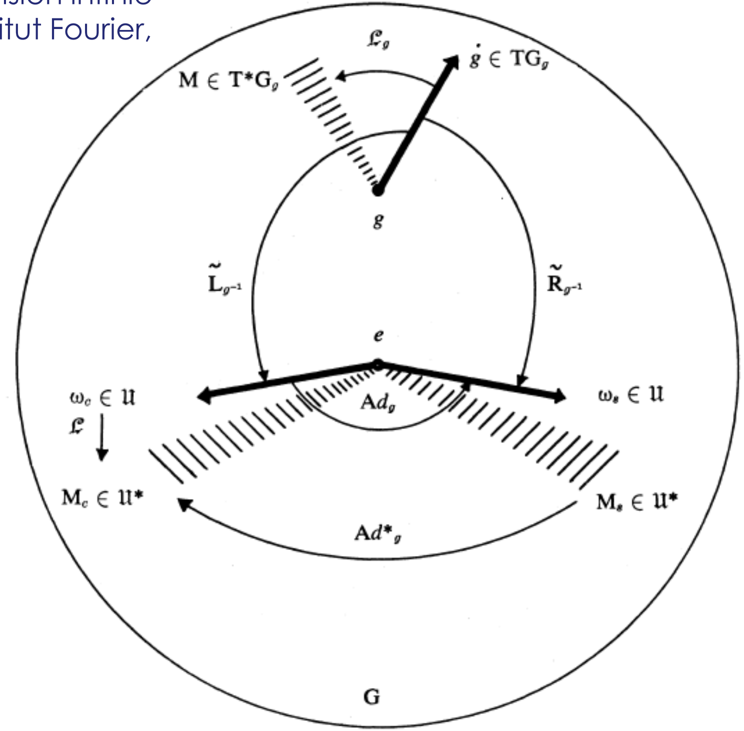

FIGURE 1.
Les vecteurs tangents à G sont représentés par des flèches droites; les vecteurs cotangents, par des séries de hachures parallèles, représentant les plans de niveau d'une 1-forme correspondante sur l'espace tangent.

From Vladimir Arnold Paper

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Affine Group Action

- Consider the General Linear Group $GL(n)$ consisting of the invertible nxn matrices, that is a topological group acting linearly on R^n by:

$$GL(n)\times R^n \to R^n$$

$$(A,x)\mapsto Ax$$

- The Group $GL(n)$ is a Lie group, is a subgroup of the General affine group $GA(n)$, composed of all pairs (A,υ) where $A\in GL(n)$ and $\upsilon\in R^n$, the group operation given by: $(A_1,\upsilon_1)(A_2,\upsilon_2)=(A_1A_2,A_1\upsilon_2+\upsilon_1)$

- Restricting A to have positive determinant one obtains the Positive General affine group $GA_+(n)$ that acts transitively on R^n by:

$$((A,\upsilon),x)\mapsto Ax+\upsilon$$

- Given a positive semidefinite matrix R, according to the spectral theorem, the continuous functional calculus can be applied to obtain a matrix $R^{1/2}$ such that $R^{1/2}$ is itself positive and $R^{1/2}R^{1/2}=R$. The operator $R^{1/2}$ is the unique non-negative square root of R.

THALES

Affine GroupAction

- $N_n = \{\aleph(\mu,\Sigma) / \mu \in R^n, \Sigma \in Sym^+{}_n\}$ the class of regular multivariate normal distributions, where μ is the mean vector and Σ is the (symmetric positive definite) covariance matrix, is invariant under the transitive action of $GA(n)$. The induced action of $GA(n)$ on $R^n \times Sym^+{}_n$ is then given by:

$$GA(n) \times \left(R^n \times Sym^+ n\right) \to R^n \times Sym^+ n$$

$$((A,\upsilon),(\mu,\Sigma)) \mapsto \left(A\mu + \upsilon, A\Sigma A^T\right)$$

- and : $GA(n) \times R^n \to R^n$

$$((A,\upsilon),x) \mapsto Ax + \upsilon$$

- As the isotropy group of $\left(0, I_n\right)$ is equal to $O(n)$, we can observe that:

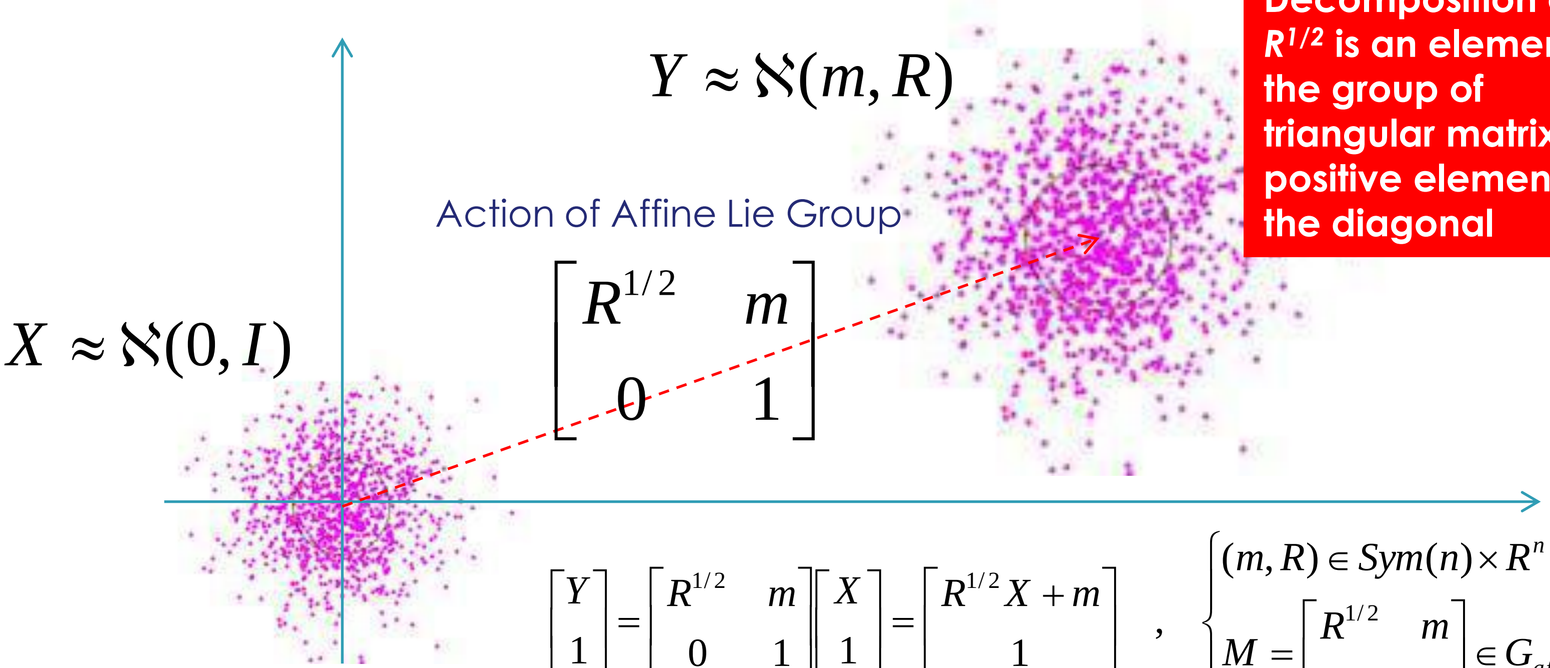
$$N_n = GA(n)/O(n)$$

- N_n is an open subset of the vector space $T_n = \{(\eta,\Omega)/\eta \in R^n, \Omega \in Sym_n\}$ and is a differentiable manifold, where the tangent space at any point may be identified with T_n.

Lie Group Everywhere: example of multivariate gaussian law

Affine group for Multivariate Gaussian Law

$R^{1/2}$: Cholesky Decomposition of R
$R^{1/2}$ is an element of the group of triangular matrix with positive elements on the diagonal



$$\begin{bmatrix} Y \\ 1 \end{bmatrix} = \begin{bmatrix} R^{1/2} & m \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix} = \begin{bmatrix} R^{1/2}X + m \\ 1 \end{bmatrix} \quad , \quad \begin{cases} (m,R) \in Sym(n) \times R^n \\ M = \begin{bmatrix} R^{1/2} & m \\ 0 & 1 \end{bmatrix} \in G_{aff} \end{cases}$$

$$X \approx \aleph(0,I) \rightarrow \aleph(m,R)$$

THALES

Affine Group (Lie Group) and associated Lie Algebra

Affine Group in case of Multivariate Gaussian case

$$\begin{bmatrix} Y \\ 1 \end{bmatrix} = \begin{bmatrix} R^{1/2} & m \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix} = \begin{bmatrix} R^{1/2}X + m \\ 1 \end{bmatrix} \quad , \quad \begin{cases} (m,R) \in R^n \times Sym(n) \\ M = \begin{bmatrix} R^{1/2} & m \\ 0 & 1 \end{bmatrix} \in G_{aff} \end{cases}$$

$$X \approx \aleph(0,I) \rightarrow Y \approx \aleph(m,R)$$

Lie Group properties

$$\left.\begin{matrix} M_1.M_2 = \begin{bmatrix} R_1^{1/2} & m_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2^{1/2} & m_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1^{1/2} R_2^{1/2} & R_1^{1/2} m_2 + m_1 \\ 0 & 1 \end{bmatrix} \\ M_2.M_1 = \begin{bmatrix} R_2^{1/2} & m_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_1^{1/2} & m_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_2^{1/2} R_1^{1/2} & R_2^{1/2} m_1 + m_2 \\ 0 & 1 \end{bmatrix} \end{matrix}\right\} \Rightarrow \begin{cases} M_1.M_2 \in G_{aff} \\ M_2.M_1 \in G_{aff} \\ M_1.M_2 \neq M_2.M_1 \\ M_1.(M_2.M_3) = (M_1.M_2).M_3 \\ M_1.I = M_1 \end{cases}$$

Affine Group (Lie Group) and associated Lie Algebra

Inverse element:

$$M = \begin{bmatrix} R^{1/2} & m \\ 0 & 1 \end{bmatrix} \Rightarrow M_R^{-1} = M_L^{-1} = M^{-1} = \begin{bmatrix} R^{-1/2} & -R^{-1/2}m \\ 0 & 1 \end{bmatrix} \in G_{aff}$$

Lie Algebra: $\mathfrak{g}$

$$L_G : \begin{cases} G_{aff} \to G_{aff} \\ M \mapsto L_M N = M.N \end{cases} \quad \text{and} \quad R_G : \begin{cases} G_{aff} \to G_{aff} \\ M \mapsto R_M N = N.M \end{cases} \qquad g = T_I(G)$$

$$\gamma(t) = \begin{bmatrix} R^{1/2}(t) & m(t) \\ 0 & 1 \end{bmatrix}, \dot{\gamma}(t) = \begin{bmatrix} \dot{R}^{1/2}(t) & \dot{m}(t) \\ 0 & 0 \end{bmatrix} \Rightarrow \Gamma_L(t) = L_{M^{-1}}(\gamma(t)) = \begin{bmatrix} R^{-1/2}R^{1/2}(t) & R^{-1/2}(m(t)-m) \\ 0 & 1 \end{bmatrix}$$

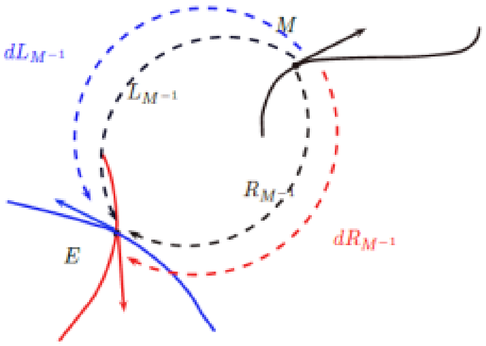
$$\dot{\Gamma}_L(t)\Big|_{t=0} = \begin{bmatrix} R^{-1/2}\dot{R}^{1/2}(0) & R^{-1/2}\dot{m}(0) \\ 0 & 1 \end{bmatrix} = \frac{d}{dt}\left(L_{M^{-1}}(\gamma(t))\right)\Big|_{t=0} = dL_{M^{-1}}\dot{\gamma}(0) = dL_{M^{-1}}\dot{M}$$

$$dL_{M^{-1}} : T_M(G) \to g_L$$

$$\dot{M} \mapsto \Omega_L = dL_{M^{-1}}\dot{M} = M^{-1}\dot{M} = \begin{bmatrix} R^{-1/2}\dot{R}^{1/2} & R^{-1/2}\dot{m} \\ 0 & 0 \end{bmatrix}$$

Affine Group (Lie Group) and associated Lie Algebra

Lie Algebra on the right and on the left



$$dL_{M^{-1}} : T_M(G) \to g_L$$

$$\dot{M} \mapsto \Omega_L = dL_{M^{-1}}\dot{M} = M^{-1}\dot{M} = \begin{bmatrix} R^{-1/2}\dot{R}^{1/2} & R^{-1/2}\dot{m} \\ 0 & 0 \end{bmatrix}$$

$$dR_{M^{-1}} : T_M(G) \to g_R$$

$$\dot{M} \mapsto \Omega_R = dR_{M^{-1}}\dot{M} = \dot{M}M^{-1} = \begin{bmatrix} R^{-1/2}\dot{R}^{1/2} & \dot{m} - R^{-1/2}\dot{R}^{1/2}\dot{m} \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} X(t) \\ 1 \end{bmatrix} = M \begin{bmatrix} x \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \dot{X}(t) \\ 0 \end{bmatrix} = \Omega_R \begin{bmatrix} X(t) \\ 1 \end{bmatrix} \quad \text{with } x \text{ fixed}$$

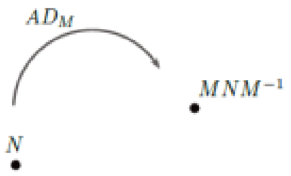
$$\begin{bmatrix} x(t) \\ 1 \end{bmatrix} = M^{-1} \begin{bmatrix} X \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \dot{x}(t) \\ 0 \end{bmatrix} = -\Omega_L \begin{bmatrix} X \\ 1 \end{bmatrix} \quad \text{with } X \text{ fixed}$$

Affine Group (Lie Group) and associated Lie Algebra

Conjugation Action

$$AD : G \times G \to G$$

$$M,N \mapsto AD_M N = M.N.M^{-1}$$

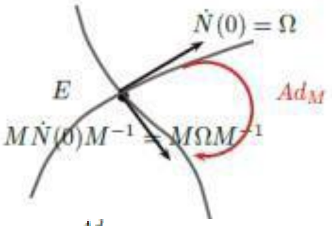


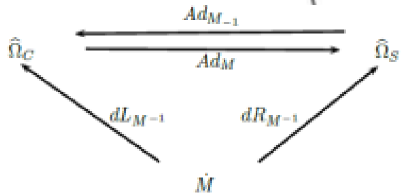
$$\begin{cases} M_1 = \begin{bmatrix} R_1^{1/2} & m_1 \\ 0 & 1 \end{bmatrix} , \quad M_2 = \begin{bmatrix} R_2^{1/2} & m_2 \\ 0 & 1 \end{bmatrix} \\ AD_{M_1} M_2 = \begin{bmatrix} R_2^{1/2} & -R_2^{1/2} m_1 + R_1^{1/2} m_2 + m_1 \\ 0 & 1 \end{bmatrix} \end{cases}$$

Adjoint Operator on Lie Group

$$Ad : G \times g \to g$$

$$M,n \mapsto Ad_M n = M.n.M^{-1} = \left. \frac{d}{dt} \right|_{t=0} \left(AD_M N(t) \right) \quad \text{with} \quad \begin{cases} N(0) = I \\ \dot{N}(0) = n \in g \end{cases}$$





$$\begin{cases} n_{2L} = \begin{bmatrix} R_2^{-1/2} \dot{R}_2^{1/2} & R_2^{-1/2} \dot{m}_2 \\ 0 & 0 \end{bmatrix}, n_{2R} = \begin{bmatrix} R_2^{-1/2} \dot{R}_2^{1/2} & -R_2^{-1/2} \dot{R}_2^{1/2} m_2 + \dot{m}_2 \\ 0 & 0 \end{bmatrix} \\ Ad_{M_1} n_{2L} = n_{2R} \quad \text{and} \quad Ad_{M_1} n_{2R} = \begin{bmatrix} R_2^{-1/2} \dot{R}_2^{1/2} & -R_2^{-1/2} \dot{R}_2^{1/2} m_2 + \dot{R}_2^{1/2} m_2 + R_2^{1/2} \dot{m}_2 \\ 0 & 0 \end{bmatrix}, Ad_{M_1^{-1}} n_{2R} = n_{2L} \end{cases}$$

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Affine Group (Lie Group) and associated Lie Algebra

Adjoint operator on Lie Algebra

$$ad : g \times g \to g$$

$$n,m \mapsto ad_m n = m.n - n.m = \left.\frac{d}{dt}\right|_{t=0} \left(Ad_M n(t)\right) = [m,n] \text{ with } \begin{cases} \dot{N}(0) = n \in g \\ \dot{M}(0) = m \in g \end{cases}$$

$$n_{1L} = \begin{bmatrix} R_1^{-1/2}\dot{R}_1^{1/2} & R_1^{-1/2}\dot{m}_1 \\ 0 & 0 \end{bmatrix}, n_{2L} = \begin{bmatrix} R_2^{-1/2}\dot{R}_2^{1/2} & R_2^{-1/2}\dot{m}_2 \\ 0 & 0 \end{bmatrix}$$

$$ad_{n_{1L}} n_{2L} = [n_{1L}, n_{2L}] = \begin{bmatrix} 0 & R_1^{-1/2}\left(\dot{R}_1^{1/2}\dot{m}_2 - \dot{R}_2^{1/2}\dot{m}_1\right)R_2^{-1/2} \\ 0 & 0 \end{bmatrix}$$

$$ad_{n_{1R}} n_{2R} = [n_{1R}, n_{2R}] = \begin{bmatrix} 0 & R_1^{-1/2}\dot{R}_1^{1/2}\left(-R_2^{-1/2}\dot{R}_2^{1/2}m_2 + \dot{m}_2\right) - R_2^{-1/2}\dot{R}_2^{1/2}\left(-R_1^{-1/2}\dot{R}_1^{1/2}m_1 + \dot{m}_1\right) \\ 0 & 0 \end{bmatrix}$$

Affine Group (Lie Group) and associated Lie Algebra

Moments Maps

$$n_L = \begin{bmatrix} R^{-1/2}\dot{R}^{1/2} & R^{-1/2}\dot{m} \\ 0 & 0 \end{bmatrix}$$

$$\Pi_L = \frac{\partial E_L}{\partial n_L} = n_L$$

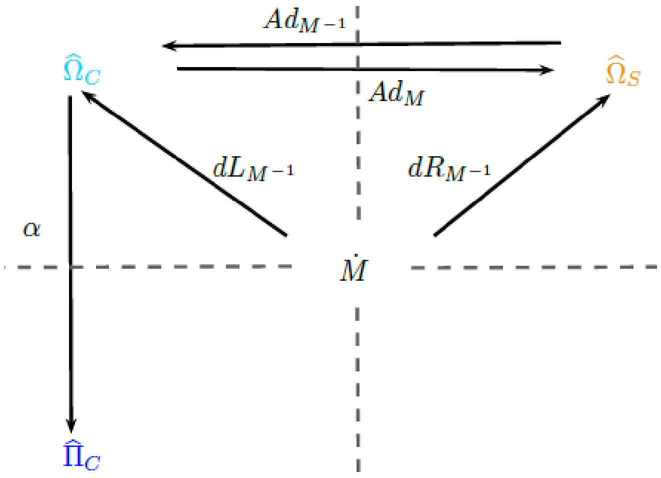
$$\langle .,. \rangle : g^* \times g \rightarrow R$$

$$k,n \mapsto \langle k,n \rangle = Tr\left(k^T n\right)$$

$$E_L = \frac{1}{2}\langle n_L, n_L \rangle = \frac{1}{2} Tr\left[n_L^T n_L\right] = \frac{1}{2}\left(Tr\left(R^{-1}\dot{R}\right) + \dot{m}^T R^{-1}\dot{m}\right)$$

$$E_L = \langle \Pi_L, n_L \rangle = \langle \Pi_L, M^{-1} n_R M \rangle = \langle \Pi_R, n_R \rangle$$

$$\Pi_L : g \rightarrow g^*$$
$$n_L \mapsto \Pi_L$$



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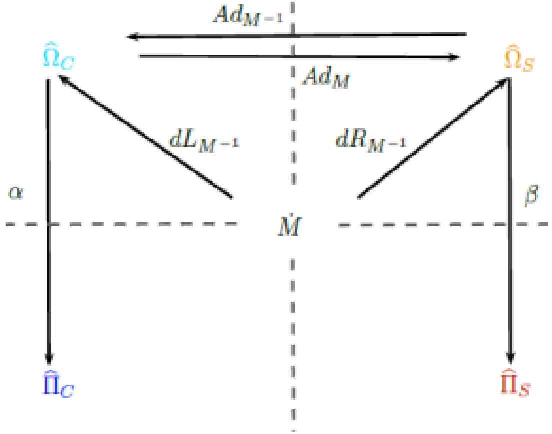
Affine Group (Lie Group) and associated Lie Algebra

Moment maps

$$E_L = \langle \Pi_L, n_L \rangle = \langle \Pi_L, M^{-1} n_R M \rangle = \langle \Pi_R, n_R \rangle \qquad \Pi_L = \frac{\partial E_L}{\partial n_L} = n_L \qquad n_L = \begin{bmatrix} R^{-1/2}\dot{R}^{1/2} & R^{-1/2}\dot{m} \\ 0 & 0 \end{bmatrix}$$

$$\langle n_L, M^{-1} n_R M \rangle = \langle \Pi_R, n_R \rangle$$

$$\Rightarrow \Pi_R = \begin{bmatrix} R^{-1/2}\dot{R}^{1/2} + R^{-1}\dot{m}m^T & R^{-1}\dot{m} \\ 0 & 0 \end{bmatrix}$$



Affine Group (Lie Group) and associated Lie Algebra

Co-adjoint operator

$$\begin{cases} Ad^* : G \times g^* \to g \\ \qquad M, \eta \mapsto Ad^*_M \eta \end{cases} \text{ with } \langle Ad^*_M \eta, n \rangle = \langle \eta, Ad_M n \rangle \text{ where } n \in g$$

$$\begin{cases} M = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} \in G \\ \eta = \begin{bmatrix} \eta_1 & \eta_2 \\ 0 & 0 \end{bmatrix} \in g^* \\ n = \begin{bmatrix} n_1 & n_2 \\ 0 & 0 \end{bmatrix} \in g \end{cases} \Rightarrow \begin{cases} \langle Ad^*_M \eta, n \rangle = \langle \eta, Ad_M n \rangle = \langle \eta, MnM^{-1} \rangle \\ \langle Ad^*_M \eta, n \rangle = \left\langle \begin{bmatrix} \eta_1 - \eta_2 b^T & A\eta_2 \\ 0 & 0 \end{bmatrix}, n \right\rangle \end{cases} \Rightarrow Ad^*_M \eta = \begin{bmatrix} \eta_1 - \eta_2 b^T & A\eta_2 \\ 0 & 0 \end{bmatrix}$$

$$Ad^*_{M^{-1}} \eta = \begin{bmatrix} \eta_1 + A\eta_2 b^T & A\eta_2 \\ 0 & 0 \end{bmatrix}$$

Affine Group (Lie Group) and associated Lie Algebra

Co-adjoint operator

$$\begin{cases} ad^* : g \times g^* \to g^* \\ \qquad n, \eta \mapsto ad_n^* \eta \end{cases} \text{ with } \langle ad_n^* \eta, \kappa \rangle = \langle \eta, ad_n \kappa \rangle \text{ where } \kappa \in g$$

$$\begin{cases} \kappa = \begin{bmatrix} \kappa_1 & \kappa_2 \\ 0 & 0 \end{bmatrix} \in G \\ \eta = \begin{bmatrix} \eta_1 & \eta_2 \\ 0 & 0 \end{bmatrix} \in g^* \\ n = \begin{bmatrix} n_1 & n_2 \\ 0 & 0 \end{bmatrix} \in g \end{cases} \Rightarrow \begin{cases} \langle ad_n^* \eta, \kappa \rangle = \langle \eta, ad_n \kappa \rangle = \langle \eta, n\kappa - \kappa n \rangle \\ \langle ad_n^* \eta, \kappa \rangle = \left\langle \begin{bmatrix} -\eta_2 n_2^T & n_1 \eta_2 \\ 0 & 0 \end{bmatrix}, \kappa \right\rangle \end{cases} \Rightarrow \begin{cases} ad_n^* \eta = \begin{bmatrix} -\eta_2 n_2^T & n_1 \eta_2 \\ 0 & 0 \end{bmatrix} \\ ad_n^* \eta = \{ n, \eta \} \end{cases}$$

Affine Group (Lie Group) and associated Lie Algebra

Moment associated to $\dot{M} \in T_M G$

$$\begin{cases} \langle \Pi_L, n_L \rangle = \langle dL^*_{M^{-1}} \Pi_L, \dot{M} \rangle \\ \langle \Pi_L, dL_{M^{-1}} \dot{M} \rangle = \langle \Pi_L, M^{-1} \dot{M} \rangle \end{cases} \Rightarrow p = \left(M^{-1}\right)^T \Pi_L$$

$$dL^*_{M^{-1}} : g^*_L \to T^*_M G$$
$$\Pi_L \mapsto p = \left(M^{-1}\right)^T \Pi_L$$

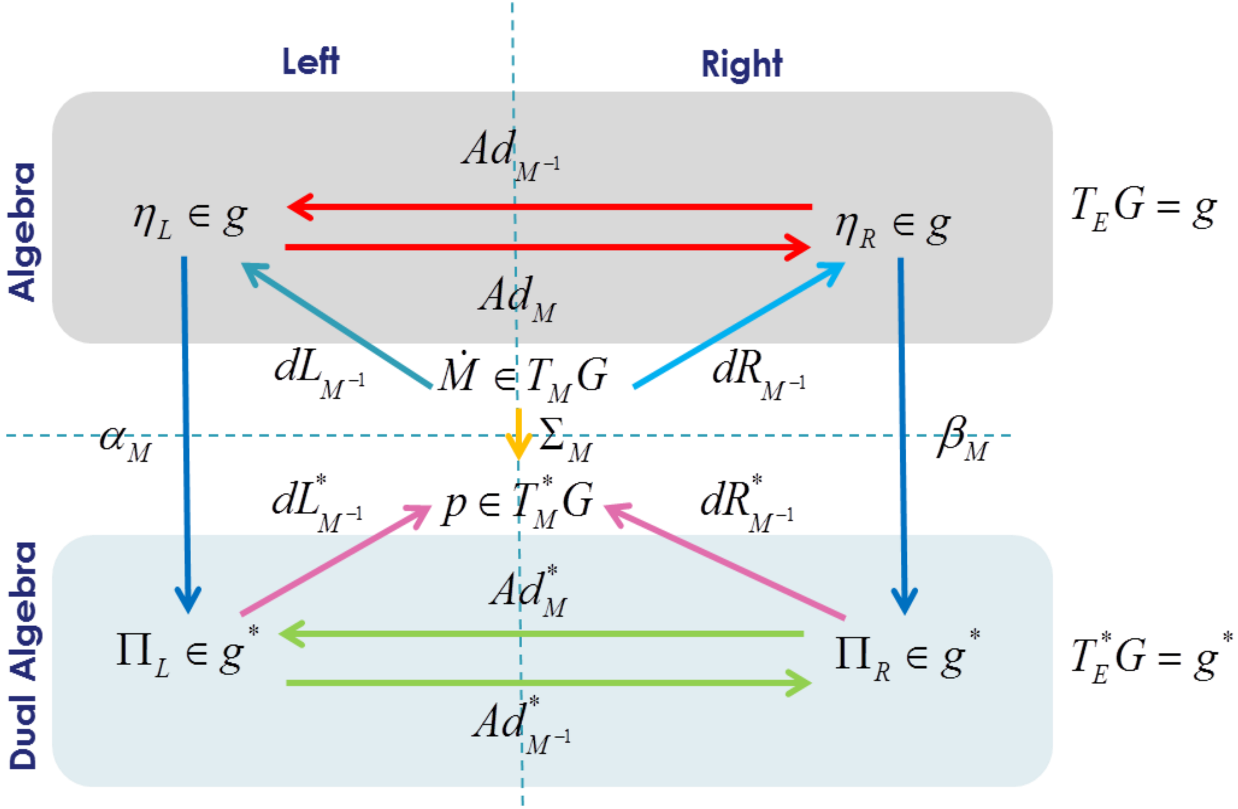
$$dR^*_{M^{-1}} : g^*_R \to T^*_M G$$
$$\Pi_R \mapsto p = \left(M^{-1}\right)^T \Pi_L$$

$$\Pi_L = n_L = M^{-1} \dot{M}$$

$$\Rightarrow \begin{cases} p = \left(M^{-1}\right)^T M^{-1} \dot{M} \\ p = \Xi_M . \dot{M} \quad \text{with} \quad \Xi_M = \left(M^{-1}\right)^T M^{-1} \end{cases}$$

$$Ad^*_M \Pi_R = \Pi_L \qquad Ad^*_{M^{-1}} \Pi_L = \Pi_R$$

Short Summary



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Affine Group (Lie Group) and associated Lie Algebra

1st Euler-Poincaré Equation:

$$\Pi_L = M^{-1}\dot{M} = \frac{\partial E_L}{\partial n_L} \in g_L \qquad S(\eta_L) = \int_a^b l(\eta_L)dt \quad \text{with} \quad \delta S(\eta_L) = 0 \quad \text{and} \quad l : g \to R$$

$$\frac{d\Pi_L}{dt} = ad^*_{n_L} \Pi_L \qquad \frac{d}{dt}\frac{\delta l}{\delta \eta_L} = ad^*_{\eta_L} \frac{\delta l}{\delta \eta_L}$$

$$\frac{\delta l}{\delta \eta_L} = \frac{\partial E_L}{\partial n_L} = \Pi_L \in g_L \qquad \delta \eta_L = \dot{\Gamma} + ad_{\eta_L} \Gamma \quad \text{where} \quad \Gamma(t) \in g$$

2nd Euler-Poincaré Equation:

$$\frac{d\Pi_R}{dt} = 0$$

Remark:

$$\begin{cases} \dot{\Pi}_{L1} = -\Pi_{L2}\eta_{L2}^T \\ \dot{\Pi}_{L2} = \eta_{L1}\Pi_{L2} \end{cases} \Rightarrow \begin{cases} \dot{\eta}_{L1} = -\eta_{L2}\eta_{L2}^T \\ \dot{\eta}_{L2} = \eta_{L1}\eta_{L2} \end{cases} \qquad \Pi_L = \frac{\partial E_L}{\partial n_L} = n_L = \begin{bmatrix} R^{-1/2}\dot{R}^{1/2} & R^{-1/2}\dot{m} \\ 0 & 0 \end{bmatrix} \qquad \begin{cases} \eta_{L1} = R^{-1/2}\dot{R}^{1/2} \\ \eta_{L2} = R^{-1/2}\dot{m} \end{cases}$$

Affine Group (Lie Group) and associated Lie Algebra

1st Euler-Poincaré Equation

$$\begin{cases}\dot{\eta}_{L1} = -\eta_{L2}\eta_{L2}^T \\ \dot{\eta}_{L2} = \eta_{L2}\eta_{L1}\end{cases} \text{ with } \begin{cases}\eta_{L1} = R^{-1/2}\dot{R}^{1/2} \\ \eta_{L2} = R^{-1/2}\dot{m}\end{cases} \Rightarrow \begin{cases}\left(R^{-1/2}\dot{R}^{1/2}\right)^{\bullet} = -R^{-1/2}\dot{m}\dot{m}^T R^{-1/2} \\ \left(R^{-1/2}\dot{m}\right)^{\bullet} = R^{-1/2}\dot{R}^{1/2}R^{-1/2}\dot{m}\end{cases}$$

2nd Euler-Poincaré equation

$$R^{-1/2}\dot{R}^{1/2} = R^{-1/2}\left(R^{-1/2}\dot{R}\right) = R^{-1}\dot{R}$$

$$\Rightarrow \Pi_R = \begin{bmatrix} R^{-1/2}\dot{R}^{1/2} + R^{-1}\dot{m}m^T & R^{-1}\dot{m} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R^{-1}\dot{R} + R^{-1}\dot{m}m^T & R^{-1}\dot{m} \\ 0 & 0 \end{bmatrix}$$

$$\frac{d\Pi_R}{dt} = \begin{bmatrix} \frac{d\left(R^{-1}\dot{R} + R^{-1}\dot{m}m^T\right)}{dt} & \frac{d\left(R^{-1}\dot{m}\right)}{dt} \\ 0 & 0 \end{bmatrix} = 0 \Rightarrow \begin{cases} R^{-1}\dot{R} + R^{-1}\dot{m}m^T = B = cste \\ R^{-1}\dot{m} = b = cste \end{cases}$$

$$\Rightarrow \begin{cases} \dot{m} = Rb \\ \dot{R} = R\left(B - bm^T\right) \end{cases}$$

Souriau theorem:
Components of moment map are invariants elements of Emmy Noether Theorem

THALES

Geodesic Computation by Euler-Poincaré Equation and Geodesic Shooting

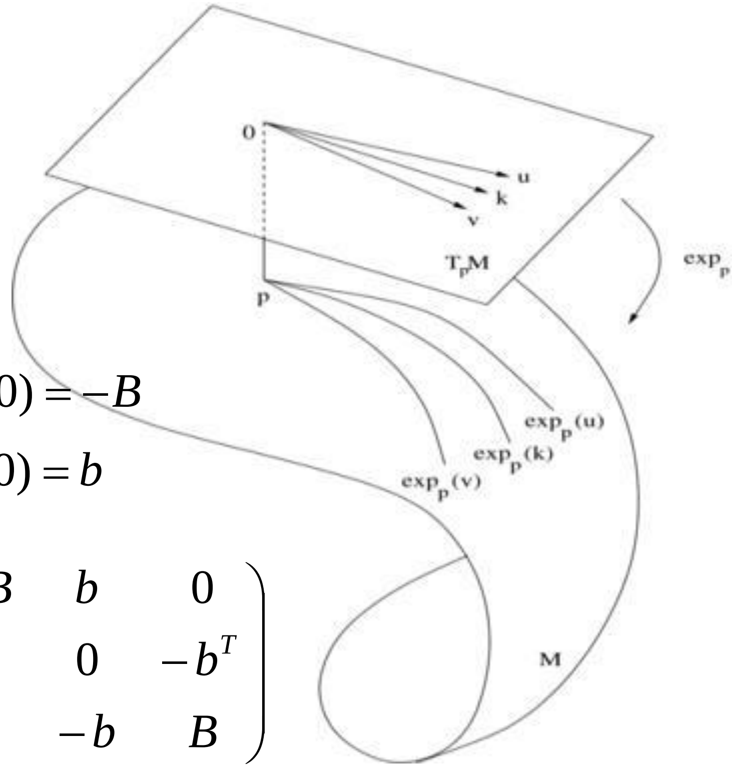
$$\text{Euler-Poincaré Equation of geodesic}\begin{cases}\dot{m}=Rb\\ \dot{R}=R\left(B-bm^T\right)\end{cases}$$

$$\text{with}\begin{cases}R^{-1}\dot{R}+R^{-1}\dot{m}m^T=B=cste\\ R^{-1}\dot{m}=b=cste\end{cases}$$

$$\begin{cases}\Delta(t)=R^{-1}(t)\\ \delta(t)=R^{-1}(t)m(t)\end{cases}\Rightarrow\begin{cases}\dot{\Delta}=-B\Delta+bm^T\\ \dot{\delta}=-B\delta+\left(1+\delta^T\Delta^{-1}\delta\right)b\\ \Delta(0)=I_p,\delta(0)=0\end{cases}\text{ with }\begin{cases}\dot{\Delta}(0)=-B\\ \dot{\delta}(0)=b\end{cases}$$

$$\Lambda(t)=\exp(tA)=\sum_{n=0}^{\infty}\frac{(tA)^n}{n!}=\begin{pmatrix}\Delta & \delta & \Phi\\ \delta^T & \varepsilon & \gamma^T\\ \Phi^T & \gamma & \Gamma\end{pmatrix}\text{ with }A=\begin{pmatrix}-B & b & 0\\ b^T & 0 & -b^T\\ 0 & -b & B\end{pmatrix}$$

$$d=\sqrt{\dot{m}(0)^T R^{-1}(0)\dot{m}(0)+\frac{1}{2}Tr\left[\left(R^{-1}(0)\dot{R}(0)\right)^2\right]}$$



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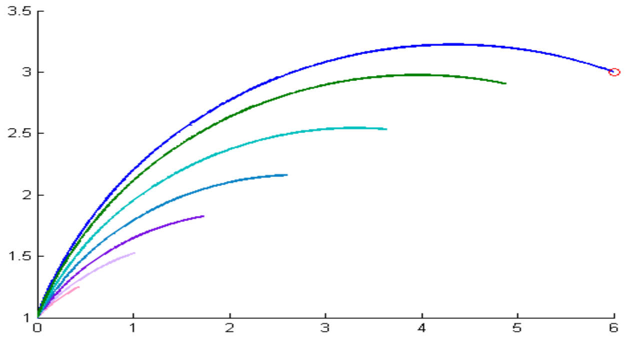
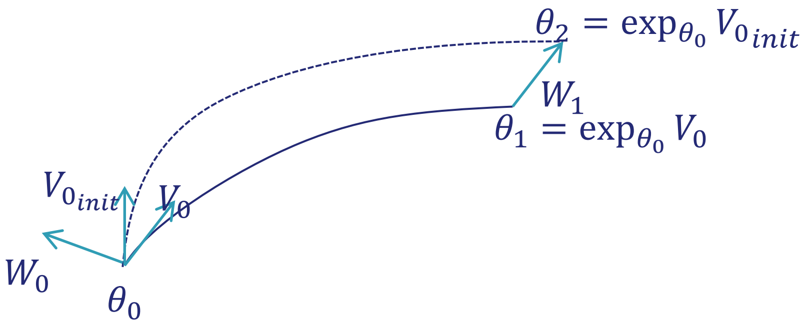
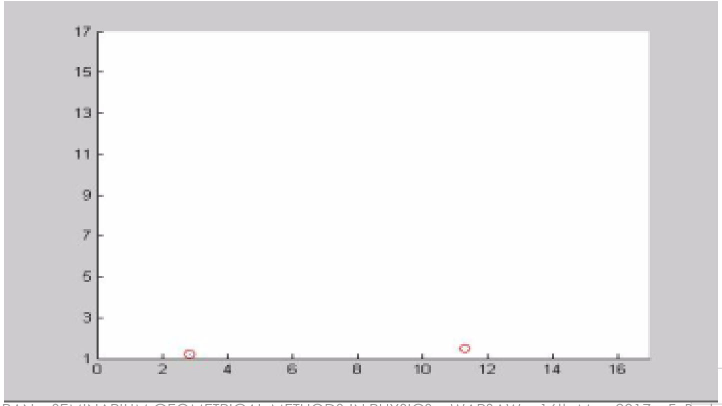
Geodesic Shooting for Multivariate Gaussian Laws (cf. Marion Pilté paper/Poster, THALES/DGA PhD student)

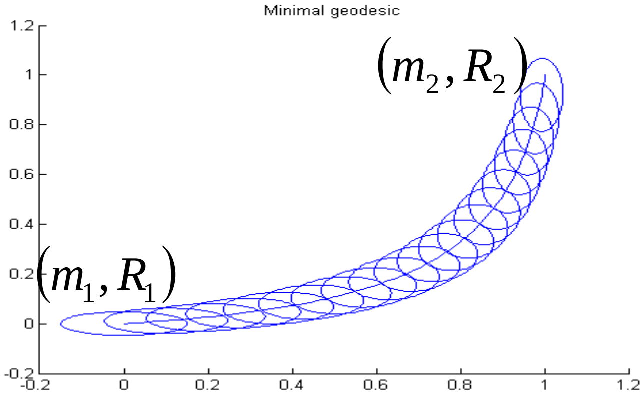




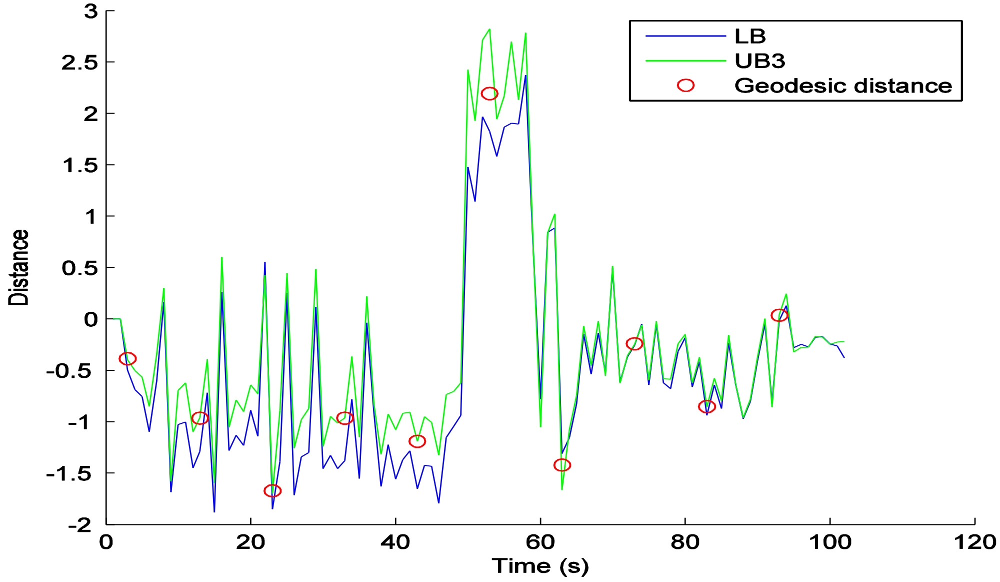
Computation of Geodesic by Geodesic Shooting based on Initial tangent vector iterative computation:

$$d = \sqrt{\dot{m}(0)^T R^{-1}(0)\dot{m}(0) + \frac{1}{2} Tr\left[\left(R^{-1}(0)\dot{R}(0)\right)^2\right]}$$





Geodesic between Multivariate Gaussian Laws for Manoeuver detection with Kalman filters



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Souriau-Fisher Metric: Example of Multivariate Gaussian Law

Variables of Multivariate Gaussian law

$$\hat{\xi}=\begin{bmatrix} E[z] \\ E[zz^T] \end{bmatrix}=\begin{bmatrix} m \\ R+mm^T \end{bmatrix},\ \beta=\begin{bmatrix} -R^{-1}m \\ \frac{1}{2}R^{-1} \end{bmatrix} \qquad Ad^*_M\hat{\xi}=\begin{bmatrix} R+mm^T-mm'^T & R'^{1/2}m \\ 0 & 0 \end{bmatrix}$$

are homeomorph to:

$$\hat{\xi}=\begin{bmatrix} R+mm^T & m \\ 0 & 0 \end{bmatrix}\in \mathrm{g},\ \beta=\begin{bmatrix} \frac{1}{2}R^{-1} & -R^{-1}m \\ 0 & 0 \end{bmatrix}$$

$$\hat{\xi}\left(Ad_M(\beta)\right)=Ad^*_M(\hat{\xi})+\theta(M)$$

$$Ad_M\beta=M.\beta.M^{-1}=\begin{bmatrix} R'^{1/2} & m' \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2}R^{-1} & -R^{-1}m \\ 0 & 0 \end{bmatrix}\begin{bmatrix} R'^{-1/2} & -R'^{-1/2}m' \\ 0 & 1 \end{bmatrix}$$

$$Ad_M\beta=\begin{bmatrix} \frac{1}{2}R'^{1/2}R^{-1}R'^{-1/2} & -\frac{1}{2}R'^{1/2}R^{-1}R'^{-1/2}m'-R'^{1/2}R^{-1}m \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} \frac{1}{2}\Omega^{-1} & -\Omega^{-1}n \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \hat{\xi}\left(Ad_M(\beta)\right)=\begin{bmatrix} \Omega+nn^T & n \\ 0 & 0 \end{bmatrix} \text{ with } \Omega=R'^{1/2}RR'^{-1/2} \text{ and } n=\left(\frac{1}{2}m'+R'^{1/2}m\right)$$

Souriau-Fisher Metric: Example of Multivariate Gaussian Law

Cohomology cycle:

$$Ad_M\beta = \begin{bmatrix} \Omega^{-1} & -\Omega^{-1}n \\ 0 & 0 \end{bmatrix} \Rightarrow \hat{\xi}(Ad_M(\beta)) = \begin{bmatrix} \Omega + nn^T & n \\ 0 & 0 \end{bmatrix} \qquad Ad_M^*\hat{\xi} = \begin{bmatrix} R + mm^T - mm'^T & R'^{1/2}m \\ 0 & 0 \end{bmatrix}$$

with $\Omega = 2R'^{1/2}\,RR'^{-1/2}$ and $n = \left(m' + 2R'^{1/2}\,m\right)$

$$\hat{\xi}(Ad_M(\beta)) = Ad_M^*(\hat{\xi}) + \theta(M) \Rightarrow \theta(M) = \hat{\xi}(Ad_M(\beta)) - Ad_M^*\hat{\xi}$$

$$= \begin{bmatrix} R'^{1/2}\,RR'^{-1/2} + \left(\frac{1}{2}m' + R'^{1/2}\,m\right)\left(\frac{1}{2}m' + R'^{1/2}\,m\right)^T & \left(\frac{1}{2}m' + R'^{1/2}\,m\right) \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} R + mm^T - mm'^T & R'^{1/2}m \\ 0 & 0 \end{bmatrix}$$

$$g_\beta([\beta,Z_1],[\beta,Z_2]) = \tilde{\Theta}_\beta(Z_1,[\beta,Z_2]) = \tilde{\Theta}(Z_1,[\beta,Z_2]) + \langle \hat{\xi}, ad_{Z_1}([\beta,Z_2]) \rangle$$

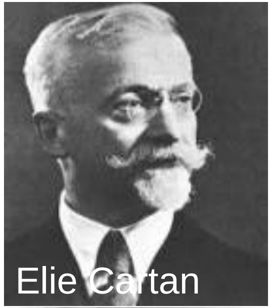
$$\tilde{\Theta}(M,Y) = \langle \Theta(M),Y \rangle = \langle T_e\theta(M),Y \rangle$$

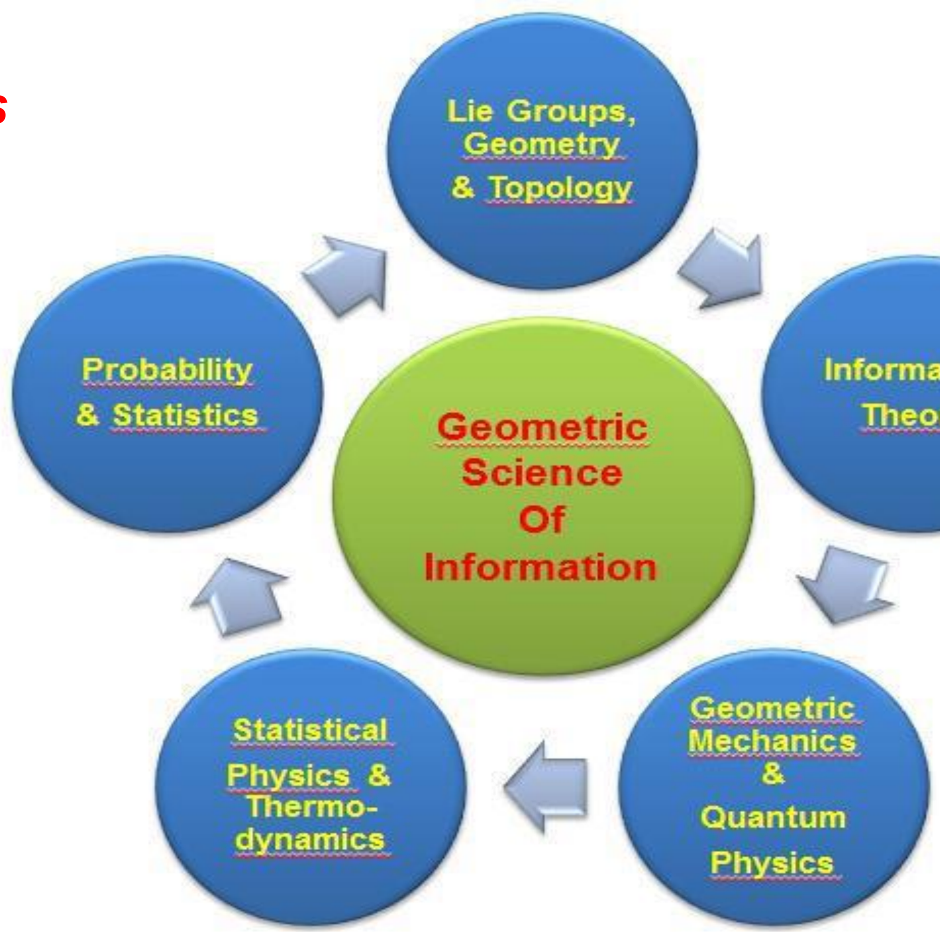
THALES

References

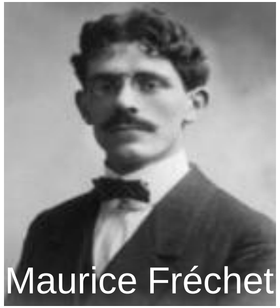
Giuseppe Longo Letmotiv : Groups & Metrics Everywhere

Groupes Omni-présents





Métriques Omni-présentes



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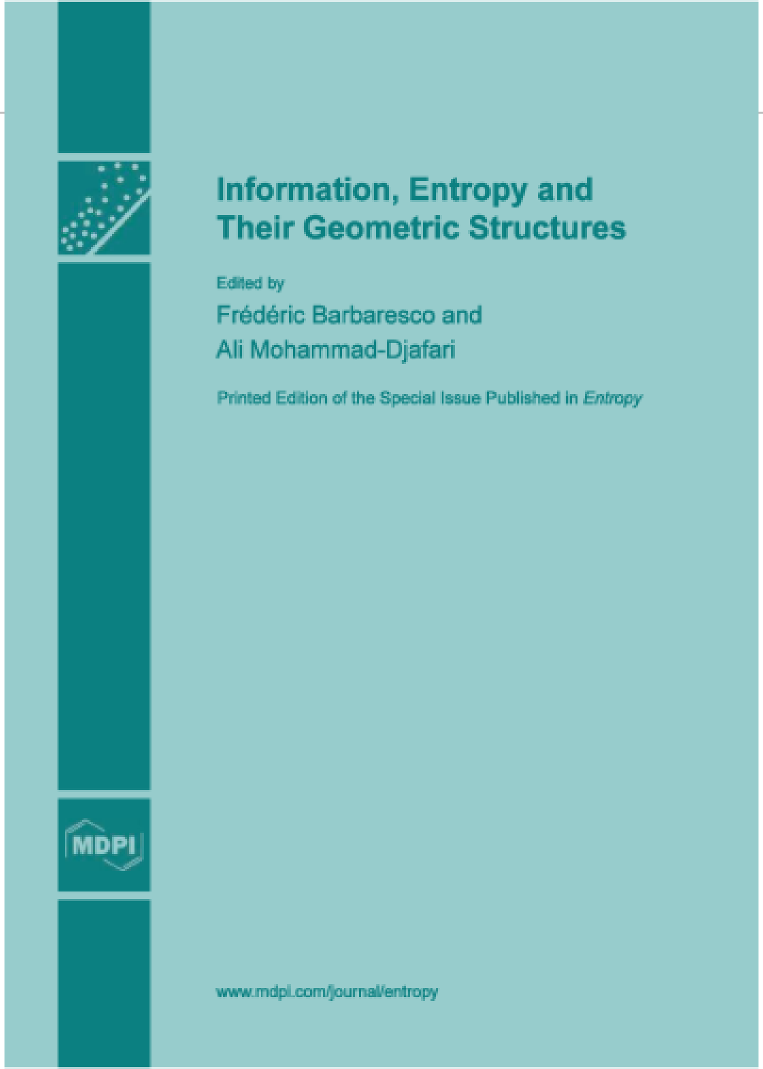

Tutorial sur la géométrie de l'Information

- Exposé de Frank Nielsen (LIX, Ecole Polytechnique) au workshop SHANNON 100 à l'Institut Henri Poincaré, fin Octobre 2016

The dual geometry of Shannon information

claude shannon
1916-2016

- Vidéo: https://www.youtube.com/watch?v=aGxZoKSk6CQ&index=11&list=PL9kd4mpdvWcDMCJ-SP72HV6Bme6CSqk_k

- Planches: https://www.lix.polytechnique.fr/~nielsen/CIG-slides.pdf

Pour aller plus loin

UNESCO UNITWIN Website on « Geometric Science of Information »:

- http://forum.cs-dc.org/category/72/geometric-science-of-information
- http://forum.cs-dc.org/topic/369/geometric-science-of-information-presentation-organisation-subscription

GSI « Geometric Science of Information » Conferences:

- GSI'15, Ecole Polytechnique: http://forum.cs-dc.org/category/90/gsi2015
- GSI'13, Ecole des Mines de Paris: https://www.see.asso.fr/gsi2013
- GSI'17 : www.si2017.org

34th International Workshop on Bayesian Inference and Maximum Entropy

- MaxEnt'14, : https://www.see.asso.fr/maxent14

Leon Brillouin Seminar on "Geometric Science of Information"

- http://repmus.ircam.fr/brillouin/home
- http://repmus.ircam.fr/brillouin/past-events

Information geometry and probability tools in abstract space for signal and image analysis

- GDR ISIS Technical Day: http://forum.cs-dc.org/topic/410/gdr-isis-gsi-day-information-geometry-and-probabilty-tools-in-abstract-space-for-signal-and-image-analysis

Google+ and Google Scholar

- https://plus.google.com/u/0/100141956413652325744/posts
- http://scholar.google.fr/citations?hl=fr&user=Pe_EE9UAAAAJ&view_op=list_works&sortby=pubdate

References

- [1] Balian, R., Alhassid, Y. and Reinhardt, H. Dissipation in many-body systems: a geometric approach based on information theory. Phys. Reports, 131, 1986, 1-146.
- [2] Balian, R., Valentin P., Hamiltonian structure of thermodynamics with gauge, Eur. Phys. J. B 21, 2001, pp. 269-282.
- [3] Balian, R. The entropy-based quantum metric, Entropy, Vol.16, n°7, 2014, pp.3878-3888.
- [4] Balian, R., François Massieu et les potentiels thermodynamiques, Évolution des disciplines et histoire des découvertes, Académie des Sciences, Avril 2015.
- [5] Barbaresco, F. Koszul information geometry and Souriau Lie group thermodynamics, AIP Conf. Proc. n°1641, n° 74, 2015, Proceedings of MaxEnt'14 conference, Amboise, Septembre 2014
- [6] Barbaresco, F. Koszul Information Geometry and Souriau Geometric Temperature/Capacity of Lie Group Thermodynamics. Entropy, vol. 16, 2014, pp. 4521-4565. Published in the book Information, Entropy and Their Geometric Structures, MDPI Publisher, September 2015.
- [7] Barbaresco, F. Symplectic Structure of Information Geometry: Fisher Metric and Euler-Poincaré Equation of Souriau Lie Group Thermodynamics. In Geometric Science of Information, Second International Conference GSI 2015 Proceedings, Lecture Notes in Computer Science vol. 9389, Springer 2015, pp. 529–540.
- [8] Barbaresco, F., Geometric Theory of Heat from Souriau Lie Groups Thermodynamics and Koszul Hessian Geometry: Applications in Information Geometry for Exponential Families. Entropy 2016, 18, 386.
- [9] Barbaresco F. , Higher Order Souriau Lie Group Thermodynamics, GSI'17 Conference, Springer LNCS, 2017
- [10] Casalis, M. Familles exponentielles naturelles invariantes par un groupe. Thèse de l'Univ. Paul Sabatier, Toulouse, 1990
- [11] Casalis, M. Familles Exponentielles Naturelles sur Rd Invariantes par un Groupe, International Statistical Review (Revue Internationale de Statistique), Vol. 59, No. 2, pp. 241-262, Aug. 1991

References

- [12] Duhem, P. Sur les équations générales de la thermodynamique, Annales scientifiques de l'ENS, 3ème série, tome 9, 1891, pp.231-266.
- [13] Eriksen, P.S. Geodesics connected with the Fisher metric on the multivariate normal manifold. Lancaster : Proceedings of the GST Workshop, 1987
- [14] Fréchet, M.R. Sur l'extension de certaines évaluations statistiques au cas de petits échantillons. Revue de l'Institut International de Statistique 1943, vol. 11, n° 3/4, pp. 182–205.
- [15] Gromov, M. In a Search for a Structure, Part 1: On Entropy, 6 July 2012, Available online: http://www.ihes.fr/~gromov/PDF/structre-serch-entropy-july5–2012.pdf (accessed on 6 August 2015).
- [16] Gromov, M. Six Lectures on Probability, Symmetry, Linearity. Available online: http://www.ihes.fr/~gromov/PDF/probability-huge-Lecture-Nov-2014.pdf (accessed on 6 August 2015).
- [17] Koszul, J.L. Sur la forme hermitienne canonique des espaces homogènes complexes. Can. J. Math. 1955, 7, 562–576.
- [18] Koszul, J.L. Exposés sur les Espaces Homogènes Symétriques; Publicação da Sociedade de Matematica de São Paulo: São Paulo, Brazil, 1959.
- [19] Koszul, J.L. Domaines bornées homogènes et orbites de groupes de transformations affines. Bull. Soc. Math. Fr. 1961, 89, 515–533.
- [20] Koszul, J.L. Ouverts convexes homogènes des espaces affines. Math. Z. 1962, 79, 254–259.
- [21] Koszul, J.L. Variétés localement plates et convexité. Osaka. J. Math. 1965, 2, 285–290.
- [22] Koszul, J.L., Introduction to Symplectic Geometry, Science Press, Beijing, 1986 (Chinese)

THALES

References

- [23] Koszul, J.L. Déformations des variétés localement plates. Ann. Inst. Fourier 1968, 18, 103–114.
- [24] Koszul, J.L. Trajectoires Convexes de Groupes Affines Unimodulaires. In Essays on Topology and Related Topics; Springer: Berlin, Germany, 1970; pp. 105–110.
- [25] Libermann, P.; Marle, C.M. Symplectic Geometry and Analytical Mechanics; Springer Science & Business Media: Berlin/Heidelberg, Germany, 1987.
- [26] Marle, C.M. Symmetries of Hamiltonian Systems on Symplectic and Poisson Manifolds, 15th Int. Conf. on Geometry, Integ. and Quant., Sofia, 2014, pp.1-86.
- [27] Marle, C.M. From Tools in Symplectic and Poisson Geometry to J.-M. Souriau's Theories of Statistical Mechanics and Thermodynamics. Entropy 2016, 18, 370.
- [28] Massieu, F. Sur les Fonctions caractéristiques des divers fluides. Comptes Rendus de l'Académie des Sciences 1869, 69, 858–862.
- [29] Massieu, F. Addition au précédent Mémoire sur les Fonctions caractéristiques. Comptes Rendus de l'Académie des Sciences 1869, 69, 1057–1061.
- [30] Massieu, F. Exposé des principes fondamentaux de la théorie mécanique de la chaleur (note destinée à servir d'introduction au Mémoire de l'auteur sur les fonctions caractéristiques des divers fluides et la théorie des vapeurs), 31 p., S.l. - s.n., 1873
- [31] Massieu, F. Thermodynamique: Mémoire sur les Fonctions Caractéristiques des Divers Fluides et sur la Théorie des Vapeurs; Académie des Sciences: Paris, France, 1876; p. 92.
- [32] Nencka, H.; Streater, R.F. Information Geometry for some Lie algebras. Infin. Dimens. Anal. Quantum Probab. Relat. Top. 1999, 2, 441–460.

References

- [33] Poincaré, H. Sur les tentatives d'explication mécanique des principes de la thermodynamique. Comptes rendus de l'Académie des sciences 1889, 108, 550–553.
- [34] Poincaré, H. Thermodynamique, Cours de Physique Mathématique; Carré, G., Ed.; 1892; en ligne: http://gallica.bnf.fr/ark:/12148/bpt6k2048983
- [35] Poincaré, H. Calcul des Probabilités; Gauthier-Villars: Paris, France, 1896.
- [36] Poincaré, H. Réflexions sur la théorie cinétique des gaz. J. Phys. Theor. Appl. 1906, 5, 369–403.
- [37] de Saxcé, G., Entropy and Structure for the Thermodynamic Systems. In Geometric Science of Information, Second International Conference GSI 2015 Proceedings, (F. Nielsen and F. Barbaresco, editors), Lecture Notes in Computer Science, vol. 9389, Springer 2015, pp. 519–528.
- [38] de Saxcé, G. Link between Lie Group Statistical Mechanics and Thermodynamics of Continua, Special Issue MDPI ENTROPY "Differential Geometrical Theory of Statistics", 2016, 18, 254
- [39] de Saxcé, G.; Vallée, C. Galilean Mechanics and Thermodynamics of Continua; Wiley-ISTE: London, UK, 2016.
- [40] Shima, H. The Geometry of Hessian Structures; World Scientific: Singapore, Singapore, 2007.
- [41] Shima, H. Geometry of Hessian Structures. In Springer Lecture Notes in Computer Science; Nielsen, F., Frederic, B., Eds.; Springer: Berlin/Heidelberg, Germany, 2013; Volume 8085, pp. 37–55.
- [42] Souriau, J.M. Géométrie de l'espace de phases. Comm. Math. Phys. 1966, 374, 1–30
- [43] Souriau, J.M. Définition covariante des équilibres thermodynamiques. Suppl. Nuov. Cimento 1966, 1, pp.203–216. (In French)

References

- [44] Souriau, J.M. Structure des systèmes dynamiques; Editions Jacques Gabay: Paris, France, 1970.
- [45] Souriau, J.-M., Mécanique statistique, groupes de Lie et cosmologie, Colloques internationaux du CNRS numéro 237, Géométrie symplectique et physique mathématique, 1974, pp. 59–113.
- [46] Souriau, J.M. Géométrie Symplectique et Physique Mathématique; Éditions du C.N.R.S.: Paris, France, 1975.
- [47] Souriau, J.M. Thermodynamique Relativiste des Fluides; Centre de Physique Théorique: Marseille, France, 1977.
- [48] Souriau, J.M. Thermodynamique et géométrie. In Differential Geometrical Methods in Mathematical Physics II; Bleuler, K., Reetz, A., Petry, H.R., Eds.; Springer: Berlin/Heidelberg, Germany, 1978; pp. 369–397.
- [49] Souriau, J.M. Dynamic Systems Structure (chap.16 convexité, chap. 17 Mesures, chap. 18 Etats statistiques, Chap. 19 Thermodynamique), available in Souriau archive (document sent by C. Vallée), unpublished technical notes, 1980.
- [50] Souriau, J.M. Mécanique classique et géométrie symplectique. CNRS Marseille. Cent. Phys. Théor. 1984, Report ref. CPT-84/PE-1695.
- [51] Souriau, J.M. On Geometric Mechanics. Discret. Cont. Dyn. Syst. J. 2007, 19, 595–607.
- [52] Vallée, C. Relativistic thermodynamics of continua. Int. J. Eng. Sci. 1981, 19, 589–601.
- [53] Vallée, C.; Lerintiu, C., Convex analysis and entropy calculation in statistical mechanics, Proceedings of A. Razmadze mathematical Institute 137, 2005, pp. 111-129.
- [54] Vey, J. Sur les Automorphismes Affines des Ouverts Convexes Saillants. Annali della Scuola Normale Superiore di Pisa, Classe di Science 1970, 24, 641–665.
- [55] Yoshizawa, S.; Tanabe, K. Dual differential geometry associated with the Kullback-Leibler information on the Gaussian distributions and its 2-parameters deformations. SUT Journal of Mathematics, vol.35, n°1, 1999, 113-137

THALES

References

- [56] Pierre Cartier, Inégalités de corrélation en mécanique statistique, Séminaire N. Bourbaki, 1972-1973, exp. N°431, p.242-264
- [57] Pierre Cartier, La durée cosmologique est-elle thermodynamique ?, séminaire ENS «Temps et durée entre science et philosophie », 2008
- [58] Pierre Cartier, Le Calcul des Probabilités de Poincaré, IHES, 2006
- [59] Moreau, J.J. Fonctions convexes duales et points proximaux dans un espace hilbertien. C. R. Acad. Sci. 1962, 255, 2897–2899
- [60] Guichardet, A. Cohomologie des Groups Topologiques et des Algèbres de Lie; Ce-dic/Fernand Nathan: Paris, France, 1980
- [61] Paul Belgodère, Extremales d'une intégrale de surface, note présentée par Elie Cartan, CRAS, séance su 11 Septembre 1944
- [62]Paulette Libermann, Legendre foliations on contact manifolds, Differential Geometry and Its Applications, n°1, pp.57-76, 1991
- [63] Paul Gauduchon, Calabi's extremal Kähler metrics: An elementary introduction, Preprint, Ecole Polytechnque
- [64] Souriau J.-M., Groupes différentiels, Lect. Notes in Math., 836, pp. 91–128. Springer Ver-lag, New-York, 1980.
- [65] Souriau J.-M., Un algorithme générateur de structures quantiques, Soc. Math. Fr., As-trisque, Actes du colloque Henri Cartan Lyon, 1985, 341–399
- [66] Souriau, J.-M., Groupes différentiels de physique mathématique, in "Feuilletages et Quan-tification Géometrique," Dazord, P. and N. Desolneux-Moulis (Eds.), Journ. lyonnaises Soc. math. France 1983, Sémin. sud-rhodanien de géom. II,Hermann, Paris, 1984; 73–119.

THALES

References

- [67] Donato P., Géometrie des orbites coadjointes des groupes de difféomorphismes, Volume 1416 of the series Lecture Notes in Mathematics , Géométrie Symplectique et Mécanique, pp 84-104, 2006
- [68] Paul Donato. Revêtement et groupe fondamental des espaces différentiels homogènes, Thèse de doctorat d'état, Université de Provence, Marseille, 1984.
- [69] Dazord P., Lie groups and algebras in infinite dimension: a new approach, Symplectic geometry and quantization, Providence, R.I. : American Mathematical Society, 1994
- [70] Hector, G., and E. Macias-Virgos, Diffeological groups, Research Exp. Math. 25 (2002), 247–260.
- [71] M. V. Losik, Fréchet manifolds as diffeological spaces, Soviet. Math. 5 (1992)
- [72] Alexandre A. Kirillov. Elements de la théorie des représentations. Editions MIR, Moscou, 1974.
- [73] Simon K. Donaldson. Moment maps and diffeomorphisms. Asian Journal of Math, vol. 3, pp. 1–16, 1999.
- [74] M. Condevaux, P. Dazord and P. Molino. Géométrie du moment. Travaux du Séminaire Sud-Rhodanien de Géométrie, I, Publ. D´ep. Math. Nouvelle Sér. B 88-1, Univ. Claude-Bernard, pp. 131 – 160, Lyon, 1988.
- [75] Augustin Banyaga. Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique, Comment. Math. Helv., volume 53, pages 174–227, 1978.
- [76] Patrick Iglesias-Zemmour, The Moment Maps in Diffeology
- [77] Helein, F., Multisymplectic geometry: some perspectives, Unfold mechanics for sound & music, IRCAM, http://medias.ircam.fr/x17b9b0

References

- [78] V. Volterra, Sulle equazioni differenziali che provengono da questiono di calcolo delle va-riazioni, Rend. Cont. Acad. Lincei, ser. IV, vol. VI, 1890, 42–54.
- [79] V. Volterra, Sopra una estensione della teoria Jacobi–Hamilton del calcolo delle varizioni, Rend. Cont. Acad. Lincei, ser. IV, vol. VI, 1890, 127–138.
- [80] J. Kijowski, A finite dimensional canonical formalism in the classical field theory, Comm. Math. Phys. 30 (1973), 99-128.
- [81] J. Kijowski, Multiphase spaces and gauge in the calculus of variations, Bull. de l'Acad. Polon. des Sci., Série sci. Math., Astr. et Phys. XXII (1974), 1219-1225.
- [82] J. Kijowski, W. Szczyrba, A canonical structure for classical field theories, Commun. Math Phys. 46 (1976), 183–206.
- [83] Jaworski W., Ingarden R.S., On the partition function in information thermodynamics with higher order temperatures, Bull. Acad.. Polon. Sci. Sér. phys. astr., pp. 28-119, 1980
- [84] Jaworski, W. On information thermodynamics with temperatures of the second order (in Polish), Master Thesis, Inst. Phys., N. Copernicus Univ., Torun, 1981
- [85] Jaworski, W., Information thermodynamics with the second order temperatures for the simplest classical systems, Acta Phys. Polon. A60, pp. 645-659., 1981
- [86] Jaworski, W., On the thermodynamic limit in information thermodynamics with higher-order temperatures, Acta Phys. Polon., A63, pp. 3-19, 1983
- [87] Jaworski, W. , Investigation of the thermodynamic limit for the states maximizing entropy under auxiliary conditions for higher-order statistical moments (in Polish). Doctor Thesis. Inst. Phys., N. Copernicus Univ. Torun, 1983
- [88] Jaworski, W., Higher-order moments and the maximum entropy inference: the thermody-namical limit approach, J. Phys. A: Math. Gen. 20, pp. 915-926, 1987

THALES

References

- [89] Ingarden, R.S. and Kossakowski A., Statistical thermodynamics with higher order temper-atures for ideal gases of bosons and fermions, Acta Phys. Polon. 28, pp.499•-511., 1965
- [90] R.S. Ingarden, A. Kossakowski, M. Ohya, Information Dynamics and Open Systems: Classical and Quantum Approach, Fundamental Theories of Physics, vol. 86, Springer, 1997
- [91] Ingarden, R.S. , Towards mesoscopic thermodynamics: small systems in higher-order states, Open Sys. Information Dynamics. 1, 75-102, 1993, errata 309.
- [92] Ingarden, R.S., and Nakagomi, T., The second order extension of the Gibbs state, Open Sys. Lnforlllalion, Jyn. 1, 259-:268., 1992
- [93] Ingarden, H.S., Meller J. , Temperatures in linguistics as a model of thermodynamics, Open Sys. Information Dynamics 2, pp.211-230, 1991
- [94] Nakagomi, T.; Mesoscopic version of thermodynamic equilibrium condition. Another ap-proach to higher order temperatures. Open systems Dyn., 1, pp. 233-2•11, 1992
- [95] J Casas-Vazquez1 and D Jou, Temperature in non-equilibrium states: a review of open problems and current proposals, Rep. Prog. Phys. 66 (2003) 1937–2023